Algebra Qualifying Exam (January 2018)

1. (*10 points*) Suppose G is an infinite group such that some noncentral element x has a finite number of conjugates. Show that G is not simple.

2. Suppose G is a group of order $455 = 5 \cdot 7 \cdot 13$.

(a) (*5 points*) Show that G has unique Sylow 7-subgroup P_7 and a unique Sylow 13-subgroup P_{13}.

(b) (*5 points*) Show that P_7 and P_{13} are contained in the center $Z(G)$.

(c) (*5 points*) Conclude that G is abelian.

3. (*10 points*) Show that $\mathbb{Z}[\sqrt{-7}]$ is not a UFD.

4. (*10 points*) Suppose R is a ring with identity and M_1 and M_2 are left R-modules. Show that M_1 and M_2 are projective if and only if $M_1 \oplus M_2$ is projective.

5. (*10 points*) Let F be a field. We say a matrix $A \in M_n(F)$ is *idempotent* if it satisfies the condition $A^2 = A$. Show that two idempotent matrices $A_1, A_2 \in M_n(F)$ are similar if and only if they have the same rank.

6. Let R be a commutative ring (with identity). Recall that an element $r \in R$ is called *nilpotent* if $r^n = 0_R$ for some positive integer n.

(a) (*6 points*) Suppose $a \in R$ is an element that is not nilpotent, and consider the multiplicative set

$$S = \{a^n \mid n \in \mathbb{Z}_{\geq 0}\}.$$

Show that the localization $S^{-1}R$ is a nonzero ring.

(b) (*6 points*) If $a \in R$ is not nilpotent, show that there exists a prime ideal $P \subset R$ such that $a \notin P$. (Hint: Take S as in part (a) and consider the localization map $f\colon R \to S^{-1}R$.)

(c) (*3 points*) Show that the set of nilpotent elements of R coincides with the intersection of all prime ideals of R.