Algebra Qualifying Exam (August 2018)

1. Let G be a group.

(a) (*5 points*) Recall that for any $g \in G$, we have the inner automorphism $\iota_g \colon G \to G$ defined by $\iota_g(x) = gxg^{-1}$, and

$$\mathrm{Inn}(G) = \{\iota_g \mid g \in G\} \subseteq \mathrm{Aut}(G)$$

is called the *group of inner automorphisms* of G. Show that $\mathrm{Inn}(G) \simeq G/Z(G)$, where $Z(G)$ is the center of G.

(b) (*3 points*) If G is abelian, show that the map $\varepsilon \colon G \to G$, $x \mapsto x^{-1}$, is an automorphism of G.

(c) (*6 points*) Show that if A is a finite abelian group of exponent 2 (i.e. every element $x \in A$ satisfies $x^2 = e$) and $|A| > 2$, then A has nontrivial automorphisms.

(d) (*6 points*) Suppose G is a finite group. Show that if G has no nontrivial automorphisms, then G has order 1 or 2.

2. Suppose G is a group of order 160 that contains two distinct subgroups H_1 and H_2 of order 80.

(a) (*5 points*) Show that $H_1 \cap H_2 \subset G$ is a normal subgroup of order 40.
(Hint: Recall that $[G : H_1 \cap H_2] = [G : H_1][H_1 : H_1 \cap H_2]$ — you may use this fact without proof.)

(b)(*10 points*) Show that G contains a normal subgroup of order 5.

3. (a) (*6 points*) Show that the ideal $I = (x^2 + 2, x^2 + 7)$ is maximal in $\mathbb{Z}[x]$.

(b) (*6 points*) Show that the polynomial $y^3 + x^2y^2 + x^3y + x$ is irreducible in $\mathbb{Z}[x, y]$.

4. (a) (*10 points*) Consider the polynomial $f(x) = x^9 - x \in \mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the field of 3 elements. Determine the number of irreducible factors and the degree of each factor in the irreducible factorization of $f(x)$ in $\mathbb{F}_3[x]$. You do not need to write down the factorization explicitly, but please provide full justification for your reasoning.

(b) (*5 points*) Let $K = \mathbb{F}_q$ be the finite field of q elements. Give an example of a polynomial $f(x, y) \in K[x, y]$ in which both variables actually appear, for which the equation $f(x, y) = 0$ has no solutions in $K \times K$. Please explain your reasoning.

5. Let R be a commutative ring with identity.

(a) (*5 points*) Suppose $P \subset R$ is a prime ideal and $I, J \subset R$ are ideals such that $P = I \cap J$. Show that $P = I$ or $P = J$.

(b) (*5 points*) Recall that an R-module M is said to be *indecomposable* if M cannot be written as $M_1 \oplus M_2$ for any nonzero submodules M_1 and M_2. Use part (a) to show that if $P \subset R$ is a prime ideal, then R/P is an indecomposable R-module.

6. (*15 points*) Let F be a field and $A \in M_n(F)$ be an $n \times n$-matrix over F. Show that A is conjugate to its transpose A^t.

7. Let $p \in \mathbb{Z}$ be a prime and consider the polynomial $f(x) = x^p - 2 \in \mathbb{Q}[x]$. Denote by $L/\mathbb{Q}$ the splitting field of $f(x)$ over $\mathbb{Q}$ (in some fixed algebraic closure of $\mathbb{Q}$).

(a) (*3 points*) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) (*6 points*) Determine the degree $[L : \mathbb{Q}]$.

(c) (*6 points*) Describe the elements of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$.

8. Let E/K is a Galois extension of fields of degree $[E : K] = 245 = 5 \cdot 7^2$. Show that

(a) (*6 points*) There exist intermediate subfields $K \subset L \subset E$ and $K \subset M \subset E$ such that $[L : K] = 5$ and $[M : K] = 49$.

(b) (*6 points*) The extensions L/K and M/K from part (a) are Galois.

(c) (*6 points*) The Galois group $\mathrm{Gal}(E/K)$ is abelian.

9. Let k be a field and consider the ring $R = k[x]/(x^2)$. Write $R = k[\varepsilon]$, where $\varepsilon^2 = 0$.
(a) (*6 points*) Show that

$$\cdots \xrightarrow{\times\varepsilon} R \xrightarrow{\times\varepsilon} R \xrightarrow{\pi} k \to 0,$$

where $\pi \colon R \to k$ is the natural surjection and the remaining maps are multiplication by ε, is a projective resolution of k as an R-module.

(b) (*7 points*) Show that $\mathrm{Hom}_{R-mod}(R, k) \simeq k$.

(c) (*7 points*) Using parts (a) and (b), show that $\mathrm{Ext}^i_R(k, k) \simeq k$ for all $i \geq 0$.