Geometry and Topology Qualifying Exam Summer 2018

This exam has **two parts**. Part I is immediately below. Part II is on page 4.

Part I

Instructions: Solve **five** of the **seven** problems below. Solve **at most two** of the **first three** problems. Do not solve more than five problems. Clearly indicate which problems you are trying to solve and which problems you are skipping.

Problem 1 (20 points).
Denote by

$$S^2 := \left\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

the 2–dimensional sphere. Define the function $h\colon\ S^2 \to \mathbf{R}$ by

$$h(x, y, z) = x + y.$$

Find the critical points of h.

Problem 2 (20 = 10 + 10 points).
Define the map $f\colon\ \mathbf{R}^4 \to \mathbf{R}^2$ by

$$f(w, x, y, z) := \begin{pmatrix} w^2 + x \\ w^2 + x^2 + y^2 + z^2 + x \end{pmatrix}.$$

Set

$$M := f^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Show that M is a submanifold of $\mathbf{R}^4$.

(b) Show that M is diffeomorphic to S^2.

Problem 3 (20 = 5 + 5 + 5 + 5 points).
Define the function $f\colon \mathbf{R}^3 \to \mathbf{R}$ by

$$f(x, y, z) := x^2 + 2y^2 - z^2.$$

(a) Determine the critical values of f.

(b) Is $f^{-1}(0)$ a submanifold of $\mathbf{R}^3$?

(c) Explain why $f^{-1}(1)$ and $f^{-1}(-1)$ are submanifolds of $\mathbf{R}^3$.

(d) How many connected components does $f^{-1}(1)$ have? How many connected components does $f^{-1}(-1)$ have?

Hint: Draw a picture for (d).

Recall: If α is a differential form of degree k and v is a vector field, then the insertion $i(v)\alpha$ is defined by $(i(v)\alpha)(v_2, \ldots, v_k) = \alpha(v, v_2, \ldots, v_k)$; also $\mathscr{L}_v\alpha$ denotes the Lie derivative of α along v.

Problem 4 (20 = 5 + 5 + 5 + 5 points).
Consider $\mathbf{R}^3$ with coordinates x, y, z. Define the 2–form α by

$$\alpha = y^2 \mathrm{d}x \wedge \mathrm{d}y + e^x \mathrm{d}y \wedge \mathrm{d}z.$$

Define the vector field v by

$$v = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Define the map $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ by

$$f(x, y, z) = (xy, yz, zx).$$

(a) Compute $\mathrm{d}\alpha$.

(b) Compute $i(v)\alpha$.

(c) Compute $\mathscr{L}_v\alpha$.

(d) Compute $f^*\alpha$.

Problem 5 (20 = 5 + 5 + 5 + 5 points).
Prove or disprove the following statements:

(a) If α is a differential form, then $\alpha \wedge \alpha = 0$.

(b) If β is a differential form and v is a vector field, then $i(v)(i(v)\beta) = 0$.

(c) If γ is a differential form and v is a vector field, then $\mathscr{L}_v \mathscr{L}_v \gamma = 0$.

(d) If f is a function, then the 1–form $\delta = f \cdot \mathrm{d}f$ is closed.

Problem 6 (20 points).
Consider $\mathbf{R}^4$ with coordinates w, x, y, z. Consider

$$T^2 = S^1 \times S^1 = \left\{ (w, x, y, z) \in \mathbf{R}^4 : w^2 + x^2 = y^2 + z^2 = 1 \right\}$$

with the product orientation of the standard orientations on S^1. Define the 2–form α on $\mathbf{R}^4$ by

$$\alpha = xyz \, \mathrm{d}w \wedge \mathrm{d}y.$$

Denote by $\iota\colon\ T^2 \to \mathbf{R}^4$ the inclusion map.

Compute the integral

$$\int_{T^2} \iota^* \alpha.$$

Problem 7 (20 = 5 + 10 + 5 points).
Let M be a manifold. Let $\pi\colon\ E \to M$ be a vector bundle of rank r.

(a) Define what it means for E to be trivial.

(b) Show that E is trivial if and only there are sections $s_1, \dots, s_r$ such that for every $x \in M$ the vectors $s_1(x), \dots, s_r(x)$ are linearly independent.

(c) Give an example of a non-trivial vector bundle.

Topology Part Qualifying Examination

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Part II: Solve **four** out of the **five** problems. Even if you attempt more than four problems, indicate which problems you want graded. You must justify your claims either by direct arguments or by referring to theorems you know.

Note. All the homology groups are with $\mathbb{Z}$-coefficients.

Problem 1. For $n \geq 0$, let S^n denote the n-dimensional sphere Prove that S^n is not homeomorphic to S^m for any $m \neq n$.

Problem 2. For $m \geq 0$ let K_m be the Klein bottle with m points removed.

(a) Calculate the fundamental group $\pi_1(K_m)$.

(b) Calculate the homology groups $H_i(K_m)$, for all $i \geq 0$.

Problem 3. Two covering spaces $p_1 : X_1 \longrightarrow X$, $p_2 : X_1 \longrightarrow X$ are called *equivalent* iff there is a homeomorphism $\tau : X_1 \longrightarrow X_2$ with $p_2 \circ \tau = p_1$.

(a) Suppose that X is path-connected and locally path-connected and that X_1, X_2 are path-connected. Let $x_1 \in X_1$ and $x_2 \in X_2$ such that $p_1(x_1) = p_2(x_2) = x_0 \in X$, for some $x_0 \in X$. If X_1, X_2 are *equivalent* by a homeomorphism $\tau : X_1 \longrightarrow X_2$ with $\tau(x_1) = (x_2)$, how is the image of $\pi_1(X_1, x_1)$ under $(p_1)_*$ related to the image of $\pi_1(X_2, x_2)$ under $(p_2)_*$? Prove your answer.

(b) Construct two *NON-equivalent* 10-sheeted covering spaces of the torus $S^1 \times S^1$. Explain why the spaces you constructed are non-equivalent.

Problem 4. Let $X := S^1 \times S^3$.

(a) Find a CW-decomposition of X and determine the corresponding cellular complex $(C_*(X),\ d)$.

(b) Compute the singular homology groups $H_i(X)$ for all i.

Problem 5. Prove or disprove the following assertions:

(a) Let M be a simply connected n-manifold. Then every sub-manifold of M is simply connected.

(b) Suppose that for connected CW-complexes X, Y we have $H_i(X) \equiv H_i(Y)$, for all $i \geq 0$. Then X and Y are homotopy equivalent.

(c) Let $Z = \mathbb{R}P^2 \vee \mathbb{R}P^2$ be the wedge sum of two projective planes. There is a covering space W of Z such that $\pi_1(W) \equiv \mathbb{Z}$. If your answer is YES give an example of such a covering space W.