MTH 868 Fall 2017: Qualifying Exam

2017-12-14

Do **six** of the eight problems below. Clearly indicate which problems you have solved by ticking the corresponding box in the following table. *Do not tick more than six boxes.*

Problem #	Solved?
1	
2	
3	
4	
5	
6	
7	
8	

Problem 1. Consider the n–dimensional torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$. Given $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, write $[x]$ for the equivalence class of x in T^n. Given an $n \times n$–matrix with integral entries $A \in \mathbf{Z}^{n\times n}$, define $f_A \colon T^n \to T^n$ by

$$f_A([x]) := [Ax].$$

Prove that for any $\omega \in \Omega^n(T^n)$ the following identity holds

$$\int_{T^n} f_A^*\omega = \det(A) \cdot \int_{T^n} \omega.$$

Problem 2. Define $f\colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$f(x, y, z) := \begin{pmatrix} x^2 + y^2 + z^2 \\ xy - z^2 \end{pmatrix}.$$

Prove that $f^{-1}(1, 0)$ is a regular submanifold of $\mathbf{R}^3$.

Problem 3. Define $\beta \in \Omega^2(\mathbf{R}^3 \backslash \{0\})$ by

$$\beta := \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{3/2}}.$$

(a) **Prove** that β is closed, that is, $\mathrm{d}\beta = 0$.

(b) **Prove** that β is not exact, that is, there is no $\alpha \in \Omega^1(\mathbf{R}^3 \backslash \{0\})$ such that $\mathrm{d}\alpha = \beta$.

Problem 4. Let M be a compact manifold. Let $E \xrightarrow{\pi} M$ be a vector bundle of rank r.
Prove that there is a natural number $n \in \mathbf{N}$ and sections $s_1, \ldots, s_n \in \Gamma(E)$ such that, for all points $x \in M$,

$$\operatorname{span}\{s_1(x), \ldots, s_n(x)\} = E_x.$$

Problem 5. On $S^1 = \mathbf{R}/2\pi\mathbf{Z}$, consider the trivial vector bundle $E = S^1 \times \mathbf{C}$. In this situation: the space of sections $\Gamma(E)$ is $C^\infty(S^1, \mathbf{C})$, the space of E–valued 1–forms $\Omega^1(M, E)$ is $\Omega^1(M, \mathbf{C})$, and the gauge group $\mathscr{G}(E)$ is $C^\infty(S^1, \mathbf{C}^*)$.

Given $\lambda \in \mathbf{C}$, define a covariant derivative ∇_λ by the following formula

$$\nabla_\lambda := d + \lambda \cdot \mathrm{d}\theta.$$

Given $\lambda, \tilde{\lambda} \in \mathbf{C}$, **prove** that ∇_λ is gauge equivalent to $\nabla_{\tilde{\lambda}}$ if and only if $\lambda - \tilde{\lambda} \in i\mathbf{Z}$.

Problem 6. **Compute** the de Rham cohomology of the open subset $U \subset \mathbf{R}^2$ given by the gray-shaded region below:

Hint: Use the Mayer–Vietoris Theorem and the homotopy invariance of de Rham cohomology.

Problem 7. Consider the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Given $\theta \in \mathbf{R}$, denote by $[\theta]$ the equivalence class of θ in S^1. Define $A\colon\ S^1 \longrightarrow \mathrm{Hom}(\mathbf{R}^3, \mathbf{R}^2)$ by

$$A([\theta]) := \begin{pmatrix} \cos(\theta) & \sin(\theta) & \sin(\theta)^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set

$$E := \left\{([\theta], v) \in S^1 \times \mathbf{R}^3 : A([\theta])v = 0\right\}$$

and define $\pi\colon\ E \longrightarrow S^1$ by

$$\pi([\theta], v) := [\theta].$$

Prove that $E \xrightarrow{\pi} S^1$ can be given the structure of a vector bundle.

Problem 8. Set

$$v := (2, 3, 5) \in \mathbf{R}^3.$$

By slight abuse of notation, we will also use v to denote the corresponding constant vector field on $\mathbf{R}^3$. Given $t \in \mathbf{R}$, define $\tau_t\colon \mathbf{R}^3 \to \mathbf{R}^3$ by

$$\tau_t(x, y, z) := (x, y, z) + tv.$$

Set

$$\Omega^k_{\text{basic}}(\mathbf{R}^3) := \left\{\alpha \in \Omega^k(\mathbf{R}^3) : i(v)\alpha = 0 \text{ and } \tau_t^*\alpha = 0 \text{ for all } t \in \mathbf{R}^3\right\}.$$

Define the twisted differential $\tilde{\mathrm{d}}\colon \Omega^\bullet(\mathbf{R}^3) \to \Omega^\bullet(\mathbf{R}^3)$ by

$$\tilde{\mathrm{d}}\alpha = \mathrm{d}\alpha - i(v)\alpha.$$

(a) **Prove** that if $\alpha \in \Omega^\bullet_{\text{basic}}(\mathbf{R}^3)$, then $\tilde{\mathrm{d}}\alpha \in \Omega^\bullet_{\text{basic}}(\mathbf{R}^3)$.

(b) **Prove** that if $\alpha \in \Omega^\bullet_{\text{basic}}(\mathbf{R}^3)$, then $\tilde{\mathrm{d}}\tilde{\mathrm{d}}\alpha = 0$.