309 Make-up Final exam 12-15-09

Name:

Throughout the exam we define the set of natural numbers by $\mathbb{N} = \{1, 2, 3, \ldots\}$**. No calculators allowed! Show all your work and justify your answers.**

(1)[14pts] Consider the set of polynomials

$$B = \{x^3 + 3x^2, x^2 + 3x, x + 3, x^3 + 4x^2 + 4x + 4\} \subseteq \mathbb{P}_3.$$

Show that B is a basis of $\mathbb{P}_3$.

(2) Let V be a finite dimensional vector space and $S, T \subseteq V$ subspaces of V. Show:

(a) [9pts] $S \cap T$ is a subspace of V.

(b) [6pts] $\dim(S \cap T) \leq \dim(T)$.

(3) Consider the set

$$S = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } \ a - b + 2c + 3d = 0 \right\} \subseteq \mathbb{R}^4.$$

(a) [9pts] Show that S is a subspace of $\mathbb{R}^4$.
(b) [3pts] Show $S \neq \mathbb{R}^4$ by exhibiting a vector in $\mathbb{R}^4$ that does not lie in S.
(c) [6pts] By direct inspection, find three vectors in S that are linearly independent. Show your list is linearly independent.
(d) [3pts] Conclude that S has dimension three.

(4) Let V be a vector space and $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\} \subseteq V$ a set of linearly independent vectors of V. Show directly: (Don't just quote a theorem!)

(a) [8pts] The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

(b) [8pts] $\mathbf{v}_{n+1} \notin \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

(5)[12pts] Prove by induction on n that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$.

(6)[16pts] Let V be an infinite dimensional vector space. Use induction to show that for every $n \in \mathbb{N}$ there is a subspace $S_n \subseteq V$ with $\dim(S_n) = n$.

(7) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear function given by:

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-2b+2c \end{bmatrix}.$$

(a) [7pts] Find the matrix of T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.
(b) [6pts] Is T one-to-one? Justify your answer!
(c) [5pts] Is T onto? Justify your answer!

(8) Let $T : \mathbb{R}^6 \longrightarrow \mathbb{M}(3, 2)$ be a linear function. Show:

(a) [10pts] If T is one-to-one, then T is onto.
(b) [10pts] If T is onto, then T is one-to-one.

(9) Let A be an $m \times n$ matrix and $\mu_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the linear function defined by $\mu_A(\mathbf{v}) = A\mathbf{v}$. Show:

(a) [10pts] The image of μ_A is a subspace of $\mathbb{R}^m$.
(b) [6pts] The vector $\mathbf{b}$ lies in the image of μ_A if and only if the non-homogenous linear system $A\mathbf{x} = \mathbf{b}$ has a solution.

(10)[16pts] Let A be an $n \times n$ matrix and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n - \{\mathbf{0}\}$ with $A\mathbf{v} = -\mathbf{v}$ and $A\mathbf{w} = 3\mathbf{w}$. Show directly that the set $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. (Don't just quote a theorem!)

(11)[16pts] Consider the ordered basis

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$ and the linear function $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by:

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a+c \\ 2a+b+c \\ b+c \end{bmatrix}.$$

Find the matrix of T with respect to the basis B.

(12) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

(a) [6pts] Find the characteristic polynomial of A.
(b) [4pts] Find the eigenvalues of A.
(c) [6pts] For every eigenvalue find an eigenvector of A.
(d) [4pts] Find an invertible matrix P so that $P^{-1}AP$ is a diagonal matrix. You do not need to verify that $P^{-1}AP$ is a diagonal matrix.