HERZOG MATH COMPETITION

November 13, 2011

Note: The grading will be similar to the grading of Putnam. This means you must explain your answers/steps as much as possible. An answer without any explanation will receive no credit.

1. While working on a problem, a student didn't notice the multiplication sign between two three-digit numbers, so she ended up writing a six digit number instead. The six digit number was seven times bigger than the product of the two three-digit numbers. Can you find the two three-digit numbers?

2. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

3. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

4. 1981 points lie inside a cube of side length 9. Prove that there are two points within a distance less than 1.

5. Prove that there are no primes in the infinite sequence of integers

$$10001, 100010001, 1000100010001, \cdots\cdots$$

6. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B anc C meet the circle again at A', B' and C' respectively. Prove that the area of triangle $A'B'C'$ is greater than or equal to the area of the triangle ABC.