Herzog Competition 2016

All problems are taken from outside sources, no claim of originality is made. The grading will be harsh, because that is how the Putnam exam is graded. So make sure you justify as much as you can every step! An answer without explanation will receive no credit. Have fun and may the algorithm be with you!

(1) The numbers a_1, a_2, a_3 are in arithmetic progression, and the numbers b_1, b_2, b_3 are in geometric progression. We know that $a_1 + b_1 = 85$, $a_2 + b_2 = 76$, $a_3 + b_3 = 84$, and $a_1 + a_2 + a_3 = 126$. Find a_1, a_2, a_3 and b_1, b_2, b_3.

(2) Determine m so that the equation in x

$$x^4 - (3m+2)x^2 + m^2 = 0$$

has four real roots in arithmetic progression.

(3) Little Marguerite is playing with little Inés, who became very good at multiplications and such (since the Herzog competition of 2014), and wants her to guess a number she wrote in a paper. So Marguerite tells Inés that the number n is the smallest natural number n which satisfies that

- its decimal representation has a 6 as its last digit, and
- if the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

How can little Inés find the number (only using mathematics and not cheating, of course)?

(4) Given a circle of radius 1 and n points $P_1, \ldots, P_n$ in the plane, prove that there is a point M on the circle such that $|MP_1| + \cdots + |MP_n| \geq n$.

(5) A rectangle in the plane with sides parallel to the x and y axes is partitioned into several smaller rectangles (which also have sides parallel to the axes). Each of the smaller rectangles has at least one side of integer length. Prove that the big rectangle has at least one side of integer length. (Note: "partition" means that the bigger rectangle is "cut" into the smaller rectangles, so that the union of all the smaller rectangles is the big rectangle, and the smaller rectangles do not overlap except possibly for touching sides or vertices.)

(6) Evaluate $\sum_{n=0}^{\infty} Arccot(n^2 + n + 1)$, where $Arccot(t)$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \frac{\pi}{2}$ with $\cot \theta = t$.