The Solutions of the Nonlinear Klein-Gordon Equation near a Steady State

by

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## 1. Introduction.

After the existence and uniqueness of solutions of the initial value problem, the structure of the solution set near equilibria is the next theoretical issue to be settled for evolution equations. We believe the picture desired here is of the qualitative dynamic properties of the solution set. In dissipative problems this structure can be determined from the linearization of the equation at the relevant equilibrium. In contrast, equations in which an energy functional is conserved are much less tractable. The nonlinear Klein-Gordon equation is conservative in nature, it conserves an energy function, and thus presents an interesting challenge.

There is a well established technique for proving the stability of an equilibrium in such conservative problems, see Strauss [10], Strauss, Shatah and Grillakis [8]. The key ingredient in this proof is that the Hessian of the energy is definite (either positive or negative). However, for the nonlinear Klein-Gordon equation on $\mathbb{R}^n$ the equilibria are not stable, see Strauss [10] and Shatah [7]. The Hessian of the energy is obviously then not definite, but this fact is not sufficient for instability, only necessary, and the proof uses other information. It appears that the Hessian is then useless for understanding the dynamics near the steady state. However, we show that it can still be used to considerably elucidate this behavior if it is coupled with the appropriate invariant manifold theorems.

For radial equilibria of the nonlinear Klein-Gordon equation in $\mathbb{R}^n$, with radial perturbations, we show that there is a finite codimension invariant manifold relative to which the equilibrium is stable and all initial data off this manifold render solutions that leave the neighborhood of the equilibrium in either forward or backward time. We use invariant manifold theorems to obtain stable, unstable and center manifolds and then apply a definite Hessian approach to the energy restricted to the center manifold. We shall set the problem up, state the lemmas and give a sketch of the proof of the main theorem; the details can be found in Bates and Jones [1].

## 2. The Klein-Gordon Equation as a Dynamical System.

The nonlinear Klein-Gordon equation which we consider here is usually written as

$$u_{tt} = \Delta u + f(u) \tag{2.1}$$

where $\Delta$ is the Laplacian in $\mathbb{R}^n$, $n \geq 3$, and $f$ is a continuously differentiable function. We shall assume that $f$ satisfies, for some positive constant $C$,

H1. $f(u) < C(1+|u|^{\gamma+1})$ where $\gamma < n/(n-2)$,

H2. $f'(u) < C(1+|u|^{\gamma})$, with the same $\gamma$,

H3. $f'(0) < 0$.

For our analysis, it is more convenient to rewrite (2.1) as a dynamical system so that the invariant manifold theory in [1] may be brought to bear upon the problem of stability of stationary solutions.



We use the same set-up as Keller [4], which dates back to Segal [6]:

$$\begin{aligned} u_t &= v \\ v_t &= \Delta u + f(u). \end{aligned} \tag{2.2}$$

It is well known that, under hypotheses H1 and H2, (2.1) has (infinitely many) radially symmetric stationary solutions (see for instance Strauss [7], Berestycki-Lions [2] or Jones-Küpper [3]). Let $\tilde{u}$ be a nontrivial solution of this type and assume that it lies in the space $H^1(\mathbb{R}^n)$. If we put $u = p + \tilde{u}$ and $v = q$ in (2.2) we obtain

$$\begin{bmatrix} p \\ q \end{bmatrix}_t = D\begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ f(p+\tilde{u}) - f(\tilde{u}) - df(\tilde{u})p \end{bmatrix} \tag{2.3}$$

where $D$ if the linearization at the stationary solution $\tilde{u} = (\tilde{u},0)$ given by

$$D = \begin{bmatrix} 0 & I \\ \Delta + df(\tilde{u}) & 0 \end{bmatrix}. \tag{2.4}$$

We consider (2.3) as defining an evolution in the space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and note that the Nemytskii operator

$$g(p,q) = (0, f(p+\tilde{u}) - f(\tilde{u}) - df(\tilde{u})p) \tag{2.5}$$

is of class $C^1$ on that Banach space and that $dg(0,0) = 0$ (see Keller [4] or Bates and Jones [1]). Also, using results in Pazy [6], one can show that $D$ generates a $C_0$-group, $S(t)$, of operators on this space. This is sufficient for the general hypotheses of our theory.

## 3. Invariant Manifolds

Here we outline some general results from [1] and indicate how they apply to the case at hand.

Consider the evolution equation

$$u_t = Au + f(u)$$

in a Banach space X. Assume

A1. A is a closed, densely defined, linear operator which generates a $C_0$-group $S(t)$ of operators on X.

A2. The spectrum of A splits into spectral sets

$$\sigma(A) = \sigma^- \cup \sigma^0 \cup \sigma^+,$$

where $\sigma^- = \{\text{Re } z < 0\}$, $\sigma^0 = \{\text{Re } z = 0\}$ and $\sigma^+ = \{\text{Re } z > 0\}$, with $\sigma^-$ and $\sigma^+$ consisting of finitely many eigenvalues each of finite multiplicity.

A3. g is of class $C^1$ on X with $dg(0) = 0$.

The first two assumptions mean that X may be decomposed as the direct sum of A-invariant closed subspaces

$$X = X^- \oplus X^0 \oplus X^+$$

such that the spectrum of $A|_{X^-}$ is $\sigma^-$, etc. We will write $S^-(t)$ for $S(t)$ restricted to $X^-$ and note that $S^-(t)$ is generated by A restricted to $X^-$. Analogous notation will be used in reference to other subspaces.

We shall also assume

A4. For all $r > 0$ there exists $M > 0$ such that

$$\|S^0(t)\| < Me^{r|t|}, \text{ for all } t.$$

<u>Remark</u>. It is tempting to assume only A1 and A2 with the hope that A4 will follow as a consequence, as it does when X is finite dimensional or when A generates an analytic semigroup. This would amount to the spectral mapping property holding, which is not the case in general and



A4 must be checked in each specific situation.

The above assumptions guarantee that equation (3.1), with initial data $u_0$ given, has a unique mild solution, existing on the interval $[-T,T]$ for some $T > 0$ (see Pazy [5]). By <u>mild solution</u> we mean a solution of the corresponding integral equation:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds.$$

Unless otherwise state, "solution" will always mean "mild solution" and in general will only be of class $C([T,T];X)$. In [1] we prove the following:

<u>Theorem 1</u>. Under the stated assumptions there exist local (near 0) invariant Lipschitz manifolds for (3.1)

$W^s$, the stable manifold, tangent to $X^-$ at 0,

$W^u$, the unstable manifold, tangent to $X^+$ at 0,

$W^c$, a center manifold, tangent to $X^0$ at 0.

<u>Remarks</u>.

1. The stable (unstable) manifold can be characterized as the set of initial conditions which give rise to solutions which decay exponentially to 0 as $t \to +(-)\infty$. Any center manifold intersects the stable and unstable manifolds only at 0 and so solutions on this manifold do not exhibit exponential decay to 0 in either forward or backward time.

The exact behavior on the center manifold requires further analysis in each individual case. In this paper we shall show that for the nonlinear Klein-Gordon equation, the base solution $\tilde{u}(r)$ is <u>stable</u> in both forward and backward time when restricted to the center manifold.

2. The local nature of these manifolds indicates that they are defined only in a neighborhood of the equilibrium $u = 0$ and invariance means relative to that neighborhood. That is, for an initial point

lying on one of these manifolds, the solution will exist, and continue to lie on that manifold, both in forward and in backward time, so long as the solution remains in the neighborhood.

3. These manifolds are represented as graphs of functions. For instance, if $V$ is an appropriate small neighborhood of $0$ in $X^-$, then there exists a Lipschitz continuous function $h^s : V \to X^0 \oplus X^+$, which is differentiable at $0$ with $dh^s(0) = 0$, such that $W^s = \{v + h^s(v) : v \in V\}$.

4. The stable and unstable manifolds are unique but center manifolds are not in general. Consequently, center-stable and center-unstable manifolds, which we will not use here but are also shown to exist in [1], are generally not unique.

Now consider the nonlinear Klein-Gordon equation as expressed by (2.3). With $D = A$ as given by (2.4) and $g$ given by (2.5), the hypotheses A1 and A3 are satisfied on $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ as is indeed commented in section 2. If the initial data in this evolution is radially symmetric, then so is the resulting solution. It follows that the subspace consisting of radial functions is invariant under this evolution; we shall denote this as $X = H^1_r(\mathbb{R}^n)\times L^2_r(\mathbb{R}^n)$.

Setting $L = \Delta + df(\tilde{u})$, the spectrum of $D$ is +/- the square root of the spectrum of $L$, see Bates and Jones [ ]. Under the hypothesis H3 and some simple Sturm-Liouville theory on $L$ restricted to $H^1_r$, the assumption A2 can be easily verified. A4 is harder to show and uses some compactness considerations.

We thus obtain by an application of Theorem 1, stable, unstable and center manifolds in a neighborhood of $\tilde{U}(r) = (\tilde{u}(r), 0)$. From the spectrum of $D$, $\sigma^+$ and $\sigma^-$ are non-empty and therefore $W^s$ and $W^u$ are both non-empty. It follows that $\tilde{U}(r)$ is unstable but its center manifold is infinite-dimensional.



4. <u>Stability on the center manifold</u>.

From the previous section $\tilde{U}$ has a local center manifold in $H^1_r\times L^2_r$. We shall show below that $\tilde{U}$ is stable with respect to the perturbations of initial conditions inside $W^c$. Let $N$ be the neighborhood of $\tilde{U}(r)$ on which the invariant manifolds are constructed, also let $h : X^0 \cap N \to X^- \oplus X^-$ be the function whose graph is $W^c$.

<u>Theorem 2</u>. There is a neighborhood $V$ of $\tilde{U}$ in $H^1_r(\mathbb{R}^n)\times L^2_r(\mathbb{R}^n)$ so that if $U_0 \in V \cap W^c$ then $U(t) \in V \cap W^c$ for all $t \in \mathbb{R}$. Moreover, if $U_0 \in V\backslash W^c$ $(W^c \neq V)$ then $U(t) \notin V$ for some $t \in \mathbb{R}$.

<u>Sketch of Proof</u>. We consider the energy $I(U)$ as defined above and calculate, if $U \in N$

$$I(\tilde{U}+U+h(U)) = I(\tilde{U}) + dI(\tilde{U})(U+h(U)) \tag{4.1}$$

$$+ \frac{1}{2}\langle d^2I(\tilde{U})(U+h(U)), U+h(U)\rangle$$

$$+ o(\|U+h(U)\|^2)$$

$$= I(\tilde{U}) + \frac{1}{2}\langle d^2I(\tilde{U})U, U\rangle + o(\|U\|^2)$$

since $h(U) = o(\|U\|)$ as $\|U\| \to 0$. Here $\langle\ ,\ \rangle$ denotes the duality pairing in $H^1\times L^2$. Now

$$\langle d^2I(\tilde{U})U, U\rangle = -(Lu, u) + \|v\|^2 \tag{4.2}$$

for each $U = (u, v) \in H^1_r\times L^2_r$ and $(\ ,\ )$ is the pairing in $L^2$.

The idea is to show that if $U \in W^c$ then the Hessian of the energy in (4.1) is positive definite. To achieve this we must relate $U \in X^c$ to $L$. Let $Y$ denote the invariant subspace of $H^1$ associated to the spectrum of $L$ in $(-\infty, 0)$ and $\pi : H^1\times L^2 \to H^1$ be the obvious projection. The crucial lemma is the following.

<u>Lemma</u>. $\pi(X^c) \subset Y$.

Remark. This lemma says that the first components of points in $X^c$ come from $Y$. If $L$ had only point spectrum in $(-\infty,0)$ the way that eigenfunctions for $D$ are constructed out of those for $L$ would make this trivial. However, with the domain being $\mathbb{R}^n$ the invariant subspaces associated with the relevant spectra of these operators are determined using the operational calculus. The proof is given in Bates and Jones [1].

In view of the lemma we have

$$-(Lu,u) \geq c_1\|u\|^2_{H^1} \tag{4.3}$$

for some $c_1 > 0$ and all $U = (u,v) \in X^c$. From (4.1), (4.2) and (4.3) it then follows that if $U \in N$

$$I(\tilde{U}+U+h(U)) - I(\tilde{U}) \geq c\|U\|^2 + O(\|U\|^2). \tag{4.4}$$

The restriction of $I$ to $W^c$ then has a strict local minimum at $\tilde{U}$.

The conservation of $I$ allows us to conclude that $\tilde{U}$ is stable in $W^c$. In particular, if $E - I(\tilde{U}) > 0$ is sufficiently small then $\{U \in W^c : I(U) < E\}$ is a bounded invariant subset of $W^c$ which contains $\tilde{U}$. Let $N^-$ and $N^+$ be small neighborhoods of $0$ in $X^-$ and $X^+$, respectively, and $N^0 = \pi(\{U \in W^c : I(U) < E\})$. Then $V = N^- \oplus N^+ \oplus N^0 + \tilde{U}$ is a neighborhood of $\tilde{U}$ that satisfies the conditions of the theorem.

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