ON SOME NONLOCAL EVOLUTION EQUATIONS ARISING IN MATERIALS SCIENCE

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Abstract. Equations for a material that can exist stably in one of two homogeneous states are derived from a microscopic or lattice viewpoint with the assumption that the evolution follows a gradient flow of the free energy with respect to some metric. Alternatively, Newtonian dynamics can be considered. The resulting lattice dynamical systems are analyzed, as are equations on the continuum where the lattice interaction energy is viewed as an approximation to a Riemann integral. These equations are lattice or nonlocal versions of the Allen-Cahn, Cahn-Hilliard, Phase-Field, or Klein-Gordon equations. Some results presented here provide for the well-posedness of the equations, while others give asymptotics or quantitative behavior of special solutions, such as traveling waves or pulses. This summarizes results previously reported in papers with co-authors Xinfu Chen, Adam Chmaj, Jianlong Han, Chunlei Zhang, and Guangyu Zhao.

1. Introduction

We view a material sample as a collection of 'atoms' occupying an n-dimensional lattice Λ.

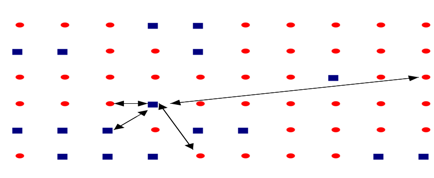

Figure 1. Lattice with long-range interactions

These atoms will be assigned 'spin' A or B but we view this as an order parameter that could represent many different things, such as true spin, local concentration or degree of solidification, etc. (when each 'atom' is really a small block of material itself). Allowing for fluctuations, we take

"We shall obtain a complete solution of the problem ... if we can express the free energy at each point as a function of the density at that point and of the differences of density in the neighboring phases, out to a distance limited by the range over which the molecular forces act."

It is possible that some molecular forces act, albeit with very small strength, at great distances and we adopt that point of view, choosing to include all pairwise interactions. The following reasoning was described in more detail in [11] and [12] but we include a brief description here for completeness.

The *Helmholtz free energy* of a state is given by

$$E = H - TS,$$

where H = interaction energy, T = absolute temperature, and S = total entropy. We include all pairwise interaction, allowing for the possibility that pairs of type A interact differently than pairs of type B and both differently from the interaction of mixed pairs. Thus,

$$H(a) \equiv -\frac{1}{2}\sum_{r,r'\in\Lambda}\Big[J^{AA}(r-r')a(r)a(r') + J^{BB}(r-r')(1-a(r))(1-a(r'))+$$

$$J^{AB}(r-r')(a(r)(1-a(r')) + a(r')(1-a(r)))\Big].$$

We expect the interaction, through the $J's$, to be symmetric and translation-invariant, but possibly anisotropic.

Rearranging:

$$H = \frac{1}{4}\sum_{r,r'\in\Lambda} J(r-r')(a(r)-a(r'))^2-$$

$$D\frac{1}{2}\sum_{r\in\Lambda}(a(r)^2 - a(r)) + d\sum_{r\in\Lambda} a(r) + \text{const.}$$

where $J(r) = J^{AA}(r) + J^{BB}(r) - 2J^{AB}(r)$, $D = \sum J(r)$, and $d = \sum(J^{BB}(r) - J^{AA}(r))/2$.

At site r the entropy $s(a(r))$ for aN particles in N identical sites is given by

$$e^{Ns/K} = \frac{N!}{(aN)!(N-aN)!}$$

where K is Boltzman's constant.

Hence,

$$s(a) \simeq -K[a\ln a + (1-a)\ln(1-a)].$$

The total entropy, $S(a) = \sum_{r\in\Lambda} s(a(r))$ and so

$$E(a) = H - TS = \frac{1}{4}\sum_{r,r'\in\Lambda} J(r-r')(a(r)-a(r'))^2+$$

$$\sum_{r\in\Lambda}\big[KT\{a(r)\ln a(r) + (1-a(r))\ln(1-a(r))\} - D(a(r)^2 - a(r)) + da(r)\big].$$

There is a critical temperature T_c such that for $T \geq T_c$ the term $[\,\cdots\,]$ is strictly convex and so there is a unique homogeneous state which minimizes $E(a)$, while for $T < T_c$, this term has two local minima and so two distinct a-states (say $\alpha < \beta$) give spatially homogeneous local minimizers of E. This is the origin of phase transition in spin systems (e.g. ferromagnets.)

Figure 2

We will fix $T < T_c$. If we were to take continuum limit by using a scaling so that the summation could be viewed as an approximation to a Riemann integral, then we would obtain a free energy in the isothermal case of the form

$$E(u) = \frac{1}{4}\iint J(x-y)(u(x)-u(y))^2 dxdy + \int F(u)dx,$$

where F is a double well function, having minima at ± 1 (after changing variables), and J is assumed to be integrable with positive integral and with $J(-x) = J(x)$.

It is interesting to compare with Ginzburg-Landau functional:

$$\int (\frac{\varepsilon^2}{2}|\nabla u|^2 + F(u))dx.$$

This is easily obtained from the above nonlocal energy by assuming the atomic interaction is short ranged so that for each state u, one could be justified by approximating $(u(x) - u(y)) \simeq (x-y)\cdot \nabla u(x)$. With that, the coefficient ε^2 in the energy is a second moment of J in the isotropic case.

In fact van der Waal's took this approach. The resulting Euler-Lagrange equation,

$$\varepsilon^2 \Delta u - F'(u) = 0,$$

has been well studied and provides some insight into phase transitions. We do not make this short-range approximation however, believing that while it may be good for a single smooth function u, it is not a good approximation in the operator sense. It is worth noting here that for several results we do not require that J be nonnegative, although it is assumed to have positive integral (or sum) and we sometimes will assume that it has a positive second moment.

Away from equilibrium we take as a fundamental principle the postulate that a material structure evolves in such a way that its free energy decreases as quickly as possible. That is, the spatial function u will evolve in such a way that $E(u)$ decreases, and does so optimally in some sense as u evolves in a function space, X. This suggests the evolution law

$$\frac{\partial u}{\partial t} = -\text{grad} E(u), \tag{1.1}$$

where grad $E(u) \in X^*$, the dual of X, is defined by

$$< \text{grad } E(u), v >= \frac{d}{dh} E(u + hv)|_{h=0}.$$

If $X = L^2$ then (1.1) becomes what we call the *nonlocal Allen-Cahn equation*,

$$\frac{\partial u}{\partial t} = J * u - Du - F'(u), \tag{1.2}$$

where $*$ is convolution and $D = \int J$ is assumed positive.

The above equation is for the case when the domain $\Omega = \mathbb{R}^n$ but for general Ω we have

$$\frac{\partial u}{\partial t} = J * u - u \int_\Omega J(x - y)dy - F'(u), \tag{1.3}$$

where $J * u(x) \equiv \int_\Omega J(x - y)u(y)dy$.

If we had made the Ginzburg-Landau approximation, the resulting gradient flow would be the Allen-Cahn equation [5],

$$\frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - F'(u).$$

Note that the operator

$$J * u - u \int_\Omega J(x - y)dy$$

may be thought of as an approximation to the Laplacian, especially in the case $J \geq 0$, since then it is a nonpositive selfadjoint operator which has a maximum principle. However, unlike the Laplacian, it is bounded and so (1.3) does not smooth in forward time and has solutions that exist locally backwards in time.

If we retain the infinite lattice model instead of moving to Riemann integrals, the equation is similar but in this discrete case convolution is given by $J * u(r) = \sum_{s \in \Lambda} J(r - s)u(s)$. For both the continuous and lattice versions, there is now a large body of work giving qualitative behavior of solutions, traveling waves, propagation failure, stability and pattern formation (see, e.g., [59], [11], [12], [15], [27], [35], [36], [37], [28], [30], [29], [10], [9], and the references therein). There is other recent work on nonlocal equations (see [71], [72], and [50]) but the earliest is perhaps that by Weinberger [78].

In the case that u represents local concentration of one species in a binary alloy, then, with the idea of conserving species, we take $X = H_0^{-1}$ (the dual of H^1 with zero mean). Then (1.1) becomes what we call the *nonlocal Cahn-Hilliard equation*

$$\frac{\partial u}{\partial t} = -\Delta(J * u - u \int_\Omega J(x - y)dy - F'(u)). \tag{1.4}$$

Of course, the original Cahn-Hilliard equations, introduced in [24], has undergone intensive study (see [39], [40], [69], [68], [75], [26], [2], [3], [13], [14], [16], and the references therein) but little has been written on the nonlocal version. To the best of our knowledge, the first was Giacomin and Liebowitz [53], [54], but more recently other results have appeared (see [52] and [32]). Here, we extend some of those results, summarizing the findings in [17] and [18] on the well posedness of (1.4) and long term behavior of solutions.

When temperature is allowed to evolve and latent heat of fusion is included in the model then the free energy, E, is often taken to be

$$\text{(1.5)}\quad \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2dxdy + \int(F(u(x)) + \frac{1}{2}\theta^2)dx,$$

where u represents degree of solidification, θ is absolute temperature, and l is a latent heat coefficient. The internal energy density is given by $e = \theta + lu$ and in order to conserve the total internal energy, $I \equiv \int e$, the simplest gradient flow is with respect to $(u,e) \in L^2 \times H_0^{-1}$. This leads to the *nonlocal phase field system*:

$$\text{(1.6)}\quad u_t = J * u - u\int_\Omega J(x-y)dy - F'(u) + l\theta,$$

$$\text{(1.7)}\quad (\theta + lu)_t = \Delta\theta.$$

The local phase-field system, introduced by Fix [47], Langer [66], and Caginalp [22], has also undergone much analysis and generalization (see, e.g., Caginalp and Fife [23], Penrose and Fife [70], Kenmochi and Kubo [60], Colli and Laurencot [33], Colli and Sprekels, [34], etc.) and still is finding many new and important applications. With hysteresis and nonlocal effects there is also the work of Krejči, Sprekels, and S. Zheng, [61], [62], [74] and some previous results in [8], all of which influenced this work on well-posedness and long term behavior of solutions for the nonlocal version (1.6), (1.7). We would also like to point out recent interesting results in [45] giving stabilization in the case of analytic nonlinearity.

Finally, we are interested in Newtonian dynamics with a force derived from the isothermal energy according to

$$\frac{\partial^2 u}{\partial t^2} = -\ \mathrm{grad}E(u).$$

In L^2 this leads to a *nonlocal wave equation*

$$\text{(1.8)}\quad \frac{\partial^2 u}{\partial t^2} = J * u - u\int_\Omega J(x-y)dy - F'(u).$$

For this equation we will establish the existence of traveling pulses with $\Omega = \mathbb{R}$ and J replaced by a large amplitude, short range kernel, namely, $\frac{1}{\varepsilon^2}j_\varepsilon$ where $j_\varepsilon(x) = \frac{1}{\varepsilon}J(\frac{x}{\varepsilon})$. Thus, we will consider

$$\text{(1.9)}\quad \frac{\partial^2 u}{\partial t^2} = \frac{1}{\varepsilon^2}(j_\varepsilon * u - u) - F'(u).$$

In the discrete case, the corresponding equation takes the form

$$\ddot{u}_n = \frac{1}{\varepsilon^2}\sum_{k=-\infty}^{\infty}\alpha_k u_{n-k} - F'(u_n), \qquad n \in \mathbb{Z} \tag{1.10}$$

where $\varepsilon > 0$ and the coefficients α_k satisfy $\sum_{k=-\infty}^{\infty}\alpha_k = 0, \quad \alpha_0 < 0, \quad \alpha_k = \alpha_{-k}, \quad \sum_{k\geq 1}\alpha_k k^2 = d > 0$. This may be viewed as a generalized lattice Klein-Gordon equation. While several studies exist for versions of lattice Klein-Gordon equations (e.g., see [25], [44], [49], [63], [73], and [76], etc.), to the best of our knowledge, there are no prior results for (1.9).

In the following three sections we outline results for the nonlocal Cahn-Hilliard, phase-field, and Klein-Gordon equations, respectively. In Section 4. we also prove existence of traveling pulses for the Klein Gordon lattice system. Since the results can seem disembodied without an idea of what lies behind, we give some of the details of the proofs, and where details are lacking, we indicate the route. Our hope is that the reader will gain an appreciation for the variety of techniques that may be brought to bear on these nonlocal evolution equations. Missing here are variational methods but the reader may turn to [10], for instance, to see that those methods may also be applied in some cases.

2. Nonlocal Cahn-Hilliard Equation

The first issue to address is whether or not (1.4) is well-posed with suitable boundary conditions. These results are to be found in more detail in papers with Jianlog Han, [17] and [18]. Since the equation is second order in space (while the usual Cahn-Hilliard equation is fourth order), only one boundary condition is expected to be necessary and sufficient for existence and uniqueness of the solution. We will therefore consider both the Dirichlet and no-flux boundary condition, the latter being more natural in the sense that species should then be conserved. Thus, we consider

$$\frac{\partial u}{\partial t} = -\Delta(J * u - u\int_\Omega J(x-y)dy - f(u)) \tag{2.1}$$

with either

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{2.2}$$

or

(2.3)
$$\frac{\partial}{\partial n}(\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)) = 0 \quad \text{on} \quad \partial\Omega,$$

where $f = F'$ is of bistable type (e.g., $f(u) = u - u^3$). This second condition of Neumann type (2.3) may look peculiar but simply states that the chemical potential has no flux across the boundary. We append the initial condition

$$u(x,0) = u_0(x), \quad \text{for} \quad x \in \bar{\Omega}.$$

We treat the Neumann problem first, discussing the main points of [17]. In order to prove the existence of a classical solution to (2.1)–(2.3) we need the initial data to satisfy the boundary condition. So we assume $u_0(x) \in C^{2+\beta,\frac{2+\beta}{2}}(\bar{\Omega})$ for some $\beta > 0$, and $u_0(x)$ satisfies the compatibility condition:

$$\frac{\partial(\int_\Omega J(x-y)dy u_0(x) - \int_\Omega J(x-y)u_0(y)dy + f(u_0))}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{2.4}$$

Rewrite the initial-boundary value problem as

$$\begin{cases} \frac{\partial u}{\partial t} = a\,(x,u)\triangle u + b(x,u,\nabla u) & \text{in} \quad \Omega,\, t>0, \\ a\,(x,u)\frac{\partial u}{\partial n} + \frac{\partial a(x)}{\partial n}u(x) - \int_\Omega \frac{\partial J(x-y)}{\partial n}u(y)dy = 0 & \text{on} \quad \partial\Omega,\, t>0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.5}$$

where

$$\begin{aligned} a(x,u) &= a(x) + f'(u), \\ a(x) &= \int_\Omega J(x-y)dy, \\ b(x,u,\nabla u) &= 2\nabla a \cdot \nabla u + f''(u)|\nabla u|^2 + u\triangle a - (\triangle J) * u. \end{aligned}$$

We assume the following conditions:

(A_1) $a(x) \in C^{2+\beta}(\bar{\Omega})$, $f \in C^{2+\beta}(\mathbb{R})$.

(A_2) There exist $c_1 > 0$, $c_2 > 0$, and $r > 0$ such that

$$a(x,u) = a(x) + f'(u) \geq c_1 + c_2|u|^{2r}.$$

(A_3) $\partial\Omega$ is of class $C^{2+\beta}$.

With regard to (A_2), note that if $a(x) + f'(u(x,t)) < 0$, for some (x,t) then there is no solution beyond that point in general, since the equation is essentially a backward heat equation. Note also that (A_2) implies

$$F(u) = \int_0^u f(s)ds \geq c_3|u|^{2r+2} - c_4 \tag{2.6}$$

for some positive constants c_3, and c_4.

For any $T > 0$, denote $Q_T = \Omega \times (0,T)$. We first establish an a priori bound for solutions of (2.1)–(2.3).

Theorem 2.1. *If $u(x,t) \in C^{2,1}(\bar{Q}_T)$ is a solution of equation (2.1)-(2.3), then*

$$\max_{Q_T} |u(x,t)| \leq \bar{C}(u_0) \tag{2.7}$$

for some constant $\bar{C}(u_0)$.

In order to prove the theorem, we need the following lemma.

Lemma 2.2. *If* $u(x,t) \in C^{2,1}(\bar{Q}_T)$ *is a solution of equation (2.1)–(2.3), then there is a constant* $C(u_0)$ *such that*

$$\sup_{0\le t\le T} ||u(\cdot,t)||_q \le C(u_0) \tag{2.8}$$

for any $q \le 2r+2$.

Proof. Let

$$E(u) = \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2 dxdy + \int F(u(x))dx. \tag{2.9}$$

Since we have a gradient flow

$$\frac{dE(u)}{dt} \le 0.$$

Therefore $E(u) \le E(u_0)$, i.e.,

$$\begin{aligned}&\frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2 dxdy + \int F(u(x))dx \\ &\quad \le \frac{1}{4}\int\int J(x-y)(u_0(x)-u_0(y))^2 dxdy + \int F(u_0(x))dx.\end{aligned}$$

From condition (A_1), (2.6), and Young's inequality, we obtain

$$\int_\Omega |u|^{2r+2} dx \le C(u_0).$$

Since this is true for any $t > 0$, we have

$$\sup_{0\le t\le T}\int_\Omega u^{2r+2} dx \le C(u_0),$$

where $C(u_0)$ does not depend on T.
Since Ω is bounded, it follows that

$$\sup_{0\le t\le T} ||u||_q \le C(u_0)$$

for any $q \le 2r+2$. □

We will prove the theorem with an iteration argument, similar to that found in [1]

Proof. For $p > 1$, multiply equation (2.1) by $u|u|^{p-1}$ and integrate over Ω, to obtain

$$\begin{aligned} \int u|u|^{p-1}u_t dx = & -\int a\,(x,u)\nabla u \cdot \nabla(u|u|^{p-1}(x))dx \\ & - \int\int \nabla J(x-y)u(x)\nabla(u|u|^{p-1}(x))dydx \\ & + \int\int \nabla J(x-y)u(y)\nabla(u|u|^{p-1}(x))dydx. \end{aligned} \tag{2.10}$$

Since

$$\int_\Omega a(x,u)\nabla u \cdot \nabla(u|u|^{p-1})dx = p\int_\Omega a\,(x,u)|u|^{p-1}|\nabla u|^2 dx \tag{2.11}$$

and

$$|\nabla|u|^{\frac{p+1}{2}}|^2 = \frac{(p+1)^2}{4}|u|^{p-1}|\nabla u|^2, \tag{2.12}$$

with condition (A_2), we have

$$\begin{aligned} \int_\Omega a\,(x,u)\nabla u \cdot \nabla(u|u|^{p-1})dx \geq & \frac{4pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx \\ & + \frac{4pc_2}{(p+2r+1)^2}\int_\Omega |\nabla|u|^{\frac{p+2r+1}{2}}|^2 dx. \end{aligned} \tag{2.13}$$

This yields

$$\begin{aligned} \frac{1}{p+1}\frac{d}{dt}\int_\Omega |u|^{p+1}dx + \frac{4pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx & \\ \leq -\int\int \nabla J(x-y)u(x)\nabla(u|u|^{p-1}(x))dydx & \\ + \int\int \nabla J(x-y)u(y)\nabla(u|u|^{p-1}(x))dydx. & \end{aligned} \tag{2.14}$$

From Cauchy-Schwartz and Young's inequalities we have

$$\begin{aligned} & -\int\int \nabla J(x-y)u(x)\nabla(u|u|^{p-1}(x))dydx \\ & \leq \frac{c_1 p}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx + M_2 p\int_\Omega |u|^{p+1}dx, \end{aligned} \tag{2.15}$$

for some positive constant M_2 which does not depend on p, and $M_1 = sup \int J(x-y)dy$. Also we have

$$\begin{aligned}
&\int\int |\nabla J(x-y)||u(y)|\nabla(u|u|^{p-1}(x))dydx \\
(2.16)\qquad &= p\int\int |\nabla J(x-y)||u(y)||u(x)|^{p-1}|\nabla u(x)|dxdy \\
&\leq \frac{c_1 p}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx + M_3 p\int_\Omega |u|^{p+1}dx
\end{aligned}$$

for some constant M_3 which does not depend on p. Inequalities (2.14)-(2.16) imply

(2.17)
$$\frac{d}{dt}\int_\Omega |u|^{p+1}dx + \frac{2pc_1}{(p+1)}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx \leq C\cdot(p+1)^2\int_\Omega |u|^{p+1}dx.$$

Now we need the following Nirenberg-Gagliado inequality,

$$||D^j v||_{L^s} \leq C_1||D^m v||^a_{L^r}||v||^{1-a}_{L^q} + C_2||v||_{L^q}, \tag{2.18}$$

where

$$\frac{j}{m} \leq a \leq 1,\ \frac{1}{s} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}. \tag{2.19}$$

In (2.18), set $s=2$, $j=0$, $r=2$, $m=1$, to get

$$||v||_2^2 \leq C_1||Dv||_2^{2a}||v||_q^{2(1-a)} + C_2||v||_q^2. \tag{2.20}$$

Let $v = |u|^{\frac{\mu_k+1}{2}}$, $\mu_k = 2^k$, $q = \frac{2(\mu_{k-1}+1)}{\mu_k+1}$, and

$$a = \frac{n(2-q)}{n(2-q)+2q} = \frac{n}{n+2+2^{2-k}}. \tag{2.21}$$

Using Young's inequality this yields

$$\int_\Omega |u|^{\mu_k+1}dx \leq \epsilon\int_\Omega |\nabla|u|^{\frac{\mu_k+1}{2}}|^2 dx + c\epsilon^{-\frac{a}{1-a}}(\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}. \tag{2.22}$$

If we set $p=\mu_k$ in (2.17) and plug (2.22) into (2.17), we obtain

(2.23)
$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |u|^{\mu_k+1}dx + \frac{2c_1\mu_k}{\mu_k+1}\int_\Omega |\nabla|u|^{\frac{\mu_k+1}{2}}|^2 dx \\
&\leq C(\mu_k+1)^2(\epsilon\int_\Omega |\nabla|u|^{\frac{\mu_k+1}{2}}|^2 dx + c\epsilon^{-\frac{a}{1-a}}(\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}).
\end{aligned}$$

Choosing $\epsilon = \frac{1}{C(\mu_k+1)^2}\cdot\frac{c_1\mu_k}{\mu_k+1}$, we have

(2.24)
$$\frac{d}{dt}\int_\Omega |u|^{\mu_k+1}dx + C_1(k)\int_\Omega |\nabla |u|^{\frac{\mu_k+1}{2}}|^2 dx \leq C_2(k)(\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}},$$

where $C_1(k) = \frac{c_1\mu_k}{\mu_k+1}$, $C_2(k) = C^{\frac{1}{1-a}} \cdot c \cdot (\frac{c_1\mu_k}{\mu_k+1})^{-\frac{a}{1-a}} \cdot (\mu_k+1)^{\frac{2}{1-a}}$.

Choosing $\epsilon = 1$ in (2.22), this and (2.24) also imply

$$\frac{d}{dt}\int_\Omega |u|^{\mu_k+1}dx + C_1(k)\int_\Omega |u|^{\mu_k+1}dx \leq C_4(k)(\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}$$

where $C_4(k) = C_2(k) + c$.

By Gronwall's inequality, we have

$$\begin{aligned}\int_\Omega |u|^{\mu_k+1}dx &\leq \int_\Omega |u_0|^{\mu_k+1}dx + \frac{C_4(k)}{C_1(k)}(\sup_{t\geq 0}\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}} \\ &\leq \delta(k) max\{M_0^{\mu_k+1}|\Omega|, (\sup_{t\geq 0}\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\},\end{aligned} \tag{2.25}$$

where $\delta(k) = c(1+\mu_k)^\alpha$, $\alpha = \frac{2}{1-a}$, and $M_0 = \sup_{x\in\Omega}|u_0|$. This implies

(2.26)
$$\begin{aligned}\int_\Omega |u|^{\mu_k+1}dx &\leq \delta(k) max\{M_0^{\mu_k+1}|\Omega|, (\sup_{t\geq 0}\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\} \\ &\leq \prod_{i=0}^{k}(|\Omega|\delta(k-i))^{\frac{\mu_k+1}{\mu_{k-i}+1}} max\{M_0^{\mu_k+1}, (\sup_{t\geq 0}\int_\Omega |u|^2 dx)^{\frac{\mu_k+1}{2}}\}.\end{aligned}$$

Since $\frac{\mu_k+1}{\mu_{k-i}+1} < 2^i$, we have

$$\begin{aligned}&\delta(k)\delta(k-1)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\delta(k-2)^{\frac{\mu_k+1}{\mu_{k-2}+1}}\cdots\delta(1)^{\frac{\mu_k+1}{2}} \\ &\leq c^{2^k-1}(2^\alpha)^{-k+2^{k+1}-2}\end{aligned} \tag{2.27}$$

and

$$|\Omega|\cdot|\Omega|^{\frac{\mu_k+1}{\mu_{k-1}+1}}\cdots|\Omega|^{\frac{\mu_k+1}{2}} \leq |\Omega|^{2^k+1}. \tag{2.28}$$

Estimates (2.26)-(2.28) and Lemma 2.2 imply

$$(\int_\Omega |u|^{\mu_k+1}dx)^{\frac{1}{\mu_k+1}} \leq C|\Omega|2^{2\alpha} max\{M_0, \sup_{t\geq 0}(\int_\Omega |u|^2 dx)^{\frac{1}{2}}\} \leq \bar{C}(u_0) \tag{2.29}$$

where $\bar{C}(u_0)$ does not depend on k. Since this is true for any k, letting $k\to\infty$ in (2.29), we have

$$||u||_\infty \leq \bar{C}(u_0),$$

and therefore,

$$\sup_{0\leq t\leq T}||u||_\infty \leq \bar{C}(u_0). \tag{2.30}$$

Since $u \in C(\bar{Q}_T)$, it follows that

$$\max_{Q_T} |u(x,t)| \le \bar{C}(u_0)$$

□

Remark 2.3. In (2.30), since $\bar{C}(u_0)$ does not depend on T, we also obtain a global bound for u whenever there is global existence of a classical solution.

Since $\max_{Q_T} |u| \le M$, after a slight modification of the proof of Theorem 7.2 in Chapter V in [65], using the equivalent form (2.5) we have

Theorem 2.4. *For any solution $u \in C^{2,1}(\bar{Q}_T)$ of equation (2.1)–(2.3) having $\max_{Q_T} |u| \le C$, one has the estimates*

$$\max_{Q_T} |\nabla u| \le K_1, \ \ |u|^{(1+\delta)}_{Q_T} \le K_2, \tag{2.31}$$

where constants K_1, K_2, and δ depend only on C, $||u_0||_{C^2(\bar{\Omega})}$ and Ω, $|\cdot|^{(1+\delta)}_{Q_T}$ is a Hölder norm in [65].

In (2.5), setting $v(x,t) = u(x,t) - u_0(x)$, we obtain the equivalent form

$$\begin{cases} \frac{\partial v}{\partial t} = \tilde{a}(x,v,u_0)\triangle v + \tilde{b}(x,v,\nabla v,u_0) & \text{in} \quad \Omega,\ t>0, \\ \\ \tilde{a}(x,v,u_0)\frac{\partial v}{\partial n} + \tilde{\psi}(x,v,u_0) = 0 & \text{on} \quad \partial\Omega,\ t>0, \\ \\ v(x,0) = 0, \end{cases} \tag{2.32}$$

where

$$\begin{aligned} &\tilde{a}(x,v,u_0) = a(x,v+u_0), \\ &\tilde{b}(x,v,\nabla v,u_0) = a(x,v+u_0)\triangle u_0 + b(x,v+u_0,\nabla(v+u_0)), \end{aligned}$$

and

$$\begin{aligned} \tilde{\psi}(x,v,u_0) = \frac{\partial a(x)}{\partial n}(v(x,t)+u_0(x)) + \tilde{a}(x,v,u_0)\frac{\partial u_0}{\partial n} \\ - \int_\Omega \frac{\partial J(x-y)}{\partial n}(v(y,t)+u_0(y))dy. \end{aligned}$$

Since (2.4) implies $\tilde{\psi}(x,0,u_0) = 0$, the compatibility condition for (2.32) is also satisfied.

Denote

$$Lv = \frac{\partial v}{\partial t} - \tilde{a}(x,v,u_0)\triangle v - \tilde{b}(x,v,\nabla v,u_0),$$

and

$$L_0 v = \frac{\partial v}{\partial t} - c_1 \triangle v,$$

where c_1 is the constant in condition (A_2).

Consider the following family of problems:

(2.33)
$$\begin{cases}\lambda Lv+(1-\lambda)L_0v=0 \qquad in\ \ Q_T,\\ \lambda(\tilde{a}(x,v,u_0)\frac{\partial v}{\partial n}+\tilde{\psi}(x,v,u_0))+(1-\lambda)(c_1(\frac{\partial v}{\partial n}))=0 \text{on}\ \partial\Omega\times[0,T],\\ v(x,0)=0.\end{cases}$$

Lemma 2.5. *If $v(x,t,\lambda)\in C^{2,1}(\bar{Q}_T)$ is a solution of (2.33), then*

$$\max_{Q_T}|v(x,t,\lambda)|\leq K, \tag{2.34}$$

where K does not depend on λ.

Proof. Since $\lambda\tilde{a}(x,v,u_0)+(1-\lambda)c_1\geq\lambda c_1+(1-\lambda)c_1=c_1>0$, the terms in (2.33) also satisfy $(A_1)-(A_2)$ and so (2.34) follows from Theorem 2.1. □

Consequently one may also conclude from Lemma 2.5 and Theorem 2.4 that:

Lemma 2.6. *If $v(x,t,\lambda)\in C^{2,1}(\bar{Q}_T)$ is a solution of equation (2.33), then*

$$\max_{Q_T}|v_x(x,t,\lambda)|\leq K_1,\quad |v(x,t,\lambda)|^{(1+\delta)}_{Q_T}\leq K_2, \tag{2.35}$$

where constants K_1, K_2, and δ do not depend on λ.

Define a Banach space

$$X=\{v(x,t)\in C^{1+\beta,\frac{1+\beta}{2}}(\bar{Q}_T):v(x,0)=0\}$$

with the usual Hölder norm.

For any function $w\in X$ satisfying conditions $max_{Q_T}|w|\leq M$ and $max_{Q_T}|w_x|\leq M_1$, we consider the following linear problem

(2.36)
$$\begin{cases}v_t-(\lambda\tilde{a}(x,w,u_0)+(1-\lambda)c_1)\triangle v+\lambda\tilde{b}(x,w,\nabla w,u_0)=0 \quad in\ \ Q_T,\\ \lambda(\tilde{a}(x,w,u_0)\frac{\partial v}{\partial n}+\tilde{\psi}(x,w,u_0))+(1-\lambda)c_1\frac{\partial v}{\partial n}=0 \quad \text{on} \quad \partial\Omega\times[0,T],\\ v(x,0)=0.\end{cases}$$

It is clear that there exists a unique solution $v(x,t,\lambda)\in C^{2+\beta,\frac{2+\beta}{2}}(\bar{Q}_T)$ of (2.36).

Define $T(w,\lambda)$ by

$$v(x,t,\lambda)=T(w,\lambda).$$

It is fairly straightforward to show that for w being in a bounded set of X, $T(w,\lambda)$ is uniformly continuous in λ, and that for any fixed λ, $T(x,\lambda)$ is continuous in X.

Since $C^{2+\beta,\frac{2+\beta}{2}}(\bar{Q}_T)\hookrightarrow C^{1+\beta,\frac{1+\beta}{2}}(\bar{Q}_T)$ is compact, we have that for any fixed λ, $T(w,\lambda)$ is a compact transformation.

The Leray-Schauder Fixed Point Theorem (see, e.g., [65]) gives the existence of a solution $v(x,t)$ of (2.32), and therefore:

Theorem 2.7. *For $u_0 \in C^{2+\beta}(\bar{\Omega})$ for some $\beta > 0$ satisfying the boundary condition (2.4), there exists a solution u to (2.1)–(2.3) with $u \in C^{2+\beta, \frac{2+\beta}{2}}(\bar{Q}_T)$.*

To continue with the well-posedness question, we have

Theorem 2.8. *(Uniqueness and continuous dependence on initial data)*
If $u_1(x,t)$ and $u_2(x,t)$ are two solutions corresponding initial data $u_{10}(x)$ and $u_{20}(x)$ of equation (2.1)–(2.3), then for some C depending only on T,

$$\sup_{0 \le t \le T} \int_\Omega |u_1 - u_2| dx \le C \int_\Omega |u_{10} - u_{20}| dx. \tag{2.37}$$

Proof. For any $\theta \in C^{2,1}(\bar{Q}_T)$ with $\frac{\partial \theta}{\partial n} = 0$ *on* $\partial\Omega$, we have

(2.38)
$$\begin{aligned}\int_\Omega u_i(x,\tau)\theta(x,\tau)dx = \int_\Omega u_i(x,0)\theta(x,0)dx + \int_0^\tau \int_\Omega (u_i\theta_t + B(x,u_i)\triangle\theta)dxdt \\ + \int_0^\tau \int_\Omega \theta \triangle J * u_i dxdt + \int_0^\tau \int_{\partial\Omega} \theta \frac{\partial J}{\partial n} * u_i dxdt,\end{aligned}$$

where $B(x,u) = a(x)u + f(u)$. Hence,

(2.39)
$$\begin{aligned}&\int_\Omega (u_1 - u_2)\theta(x,\tau)dx = \int_\Omega (u_{10} - u_{20})\theta(x,0)dx \\ &\quad + \int_0^\tau \int_\Omega (u_1 - u_2)(\theta_t + H\triangle\theta)dxdt + \int_0^\tau \int_\Omega \theta \triangle J * (u_1 - u_2)dxdt \\ &\quad + \int_0^\tau \int_{\partial\Omega} \theta \frac{\partial J}{\partial n} * (u_1 - u_2)dxdt,\end{aligned}$$

where

$$H(x,t) = \begin{cases} \frac{B(x,u_1) - B(x,u_2)}{u_1 - u_2} & \text{for} \quad u_1 \ne u_2 \\ \frac{\partial B(x,u_1)}{\partial u} & \text{for} \quad u_1 = u_2. \end{cases}$$

Let θ be the solution to the final value problem

$$\begin{cases} \frac{\partial \theta}{\partial t} = -H(x,t)\triangle\theta + \beta\theta & \text{in} \quad \Omega,\ 0 \le t \le \tau, \\ \frac{\partial \theta}{\partial n} = 0 & \text{on} \quad \partial\Omega, \\ \theta(x,\tau) = h(x), \end{cases} \tag{2.40}$$

where $h(x) \in C_0^\infty(\Omega)$, $0 \le h \le 1$ and $\beta > 0$ is a constant.
By the comparison theorem, we have

$$0 \le \theta \le e^{\beta(t-\tau)}.$$

Therefore, from (2.39) we have

$$\begin{aligned}
&\int_\Omega (u_1-u_2)h dx \\
&= \int_\Omega (u_{10}-u_{20})\theta(x,0)dx + \int_0^\tau \int_\Omega (u_1-u_2)\beta\theta dx dt \\
&+ \int_0^\tau \int_\Omega \theta \triangle J * (u_1-u_2) dx dt + \int_0^\tau \int_{\partial\Omega} \theta \frac{\partial J}{\partial n} * (u_1-u_2) dx dt.
\end{aligned} \tag{2.41}$$

Hence,

$$\begin{aligned}
&\int_\Omega (u_1-u_2)h dx \\
&\leq \int_\Omega |u_{10}-u_{20}| e^{-\beta\tau} dx + \int_0^\tau \int_\Omega |u_1-u_2| \beta e^{\beta(t-\tau)} dx dt \\
&+ C_1 \int_0^\tau \int_\Omega |u_1-u_2| e^{\beta(t-\tau)} dx dt + C_2 \int_0^\tau \int_\Omega |u_1-u_2| e^{\beta(t-\tau)} dx dt.
\end{aligned} \tag{2.42}$$

Letting $\beta \to 0$ and $h \to sign(u_1-u_2)^+$ in (2.42), we have

$$\int_\Omega (u_1-u_2)^+ dx \leq \int_\Omega |u_{10}-u_{20}| dx + C_3 \int_0^\tau \int_\Omega |u_1-u_2| dx dt. \tag{2.43}$$

Interchanging u_1 and u_2 gives

$$\int_\Omega |u_1-u_2| dx \leq \int_\Omega |u_{10}-u_{20}| dx + C_3 \int_0^\tau \int_\Omega |u_1-u_2| dx dt. \tag{2.44}$$

By Gronwall's inequality, (2.44) yields

$$\int_\Omega |u_1-u_2| dx \leq C(T) \int_\Omega |u_{10}-u_{20}| dx. \tag{2.45}$$

□

Remark 2.9. If $u_0(x) \in L^\infty(\Omega)$, we can consider weak solutions as follows:
Define

$$X = \{f(x) \in C_0^\infty(\Omega) |\ g(x) = \int_\Omega J(x-y)f(y)dy,\ g(x)|\partial\Omega = 0\}$$

$$B = \text{Closure of X in the L}^2 \text{ norm}$$

Definition 2.10. A weak solution of (2.1)–(2.3) is a function $u \in C([0,T], L^2(\Omega)) \cap L^\infty(Q_T) \cap L^2([0,T], H^1(\Omega))$, $u_t \in L^2([0,T], H^{-1}(\Omega))$, $\nabla h(x,u) \in L^2((0,T), L^2(\Omega))$ such that

$$< u_t(x,t), \psi(x) > + \int_\Omega \nabla h(x,u) \cdot \nabla \psi(x) dx \\ - \int_\Omega (\nabla J * u(\cdot, s)) \cdot \nabla \psi(x) dx = 0 \tag{2.46}$$

for all $\psi \in H^1(\Omega)$ and a.e. time $0 \le t \le T$, where $h(x,u) = a(x)u + f(u)$, $a(x) = \int_\Omega J(x-y)dy$, and

$$u(x,0) = u_0(x). \tag{2.47}$$

Theorem 2.11. *If $(A_1), (A_2)$, and (A_3) are satisfied and $u_0 \in L^\infty(\Omega) \cap B$, then there exists a unique weak solution u of (2.1)–(2.3)*

The proof is as follows: Since $u_0 \in L^\infty(\Omega) \cap B$, there exists a sequence $u_0^{(k)} \in X$ such that

$$||u_0^{(k)} - u_0||_{L^2} \to 0, \\ ||u_0^{(k)}||_\infty < C, \tag{2.48}$$

where C does not depend on k. Consider equation (2.1)–(2.3) with initial data $u_0^{(k)}$. There exists a unique classical solution $u^{(k)}$. By the energy estimate and other a priori bounds, one can find a subsequence and a weak limit u such that

$$u^{(k)} \rightharpoonup u \ \ in \quad L^2((0,T), H^1(\Omega)), \\ h(x, u^{(k)}) \rightharpoonup h(x,u) \ \ in \quad L^2((0,T), H^1(\Omega)), \\ u_t^{(k)} \rightharpoonup u_t \ \ in \quad L^2((0,T), H^{-1}(\Omega)), \tag{2.49}$$

and u satisfies equation (2.46).

Now we turn to discussing the long-term behavior of solutions in the L^p norm

We establish a nonlinear version of the *Poincaré* inequality.

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^n$ be smooth and bounded. For $p \ge 1$, there is a constant $C(\Omega)$ such that for all $u \in W^{1,2p}(\Omega)$ with $\int_\Omega u = 0$*

$$\int_\Omega |u|^{2p} dx \le C(\Omega) \int_\Omega |\nabla |u|^p|^2 dx. \tag{2.50}$$

Proof. If (2.50) is not true, there exists a sequence $\{u_k\} \subset W^{1,2p}(\Omega)$ such that

$$\int_\Omega u_k = 0, \ \ \int_\Omega |u_k|^{2p} dx > k \int_\Omega |\nabla |u_k|^p|^2 dx. \tag{2.51}$$

If $w_k = \frac{u_k}{||u_k||_{2p}}$, then it follows that

$$\int_\Omega w_k = 0, \ \int_\Omega |w_k|^{2p} dx = 1, \ \int_\Omega |\nabla |w_k|^p|^2 dx < \frac{1}{k}. \tag{2.52}$$

Therefore, there exists a subsequence (still denoted by $\{|w_k|^p\}$) and $w \in H^1(\Omega)$ such that

$$|w_k|^p \rightharpoonup w \ in \ H^1 \ and \ |w_k|^p \to w \ \ in \ \ L^2. \tag{2.53}$$

Since $\int_\Omega |\nabla |w_k|^p|^2 dx \leq \frac{1}{k}$, for any $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \frac{\partial |w_k|^p}{\partial x_i} \varphi dx \to 0 \tag{2.54}$$

for i=1,$\cdots$,n. Therefore,

$$\int_\Omega \frac{\partial w}{\partial x_i} \varphi dx = 0 \tag{2.55}$$

for i=1,$\cdots$,n and $\varphi \in C_0^\infty(\Omega)$. So $\nabla w = 0$ a.e in Ω, and w is constant in Ω.

By taking a subsequence, (2.52) and (2.53) yield

$$w = (\frac{1}{|\Omega|})^{\frac{1}{2}}, \ and \ |w_k|^p \to (\frac{1}{|\Omega|})^{\frac{1}{2}} \ \ a.e \ \ in \ \Omega. \tag{2.56}$$

So, we have

$$|w_k| \to (\frac{1}{|\Omega|})^{\frac{1}{2p}} \ \ a.e \ \ in \ \Omega. \tag{2.57}$$

Since $\int w_k = 0$, there exists a unique solution φ_k to

$$\begin{cases} -\triangle \varphi = w_k & in \ \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & on \ \partial\Omega, \\ \int_\Omega \varphi dx = 0. \end{cases} \tag{2.58}$$

From (2.58), we obtain

$$\int |\nabla \varphi_k|^2 = \int w_k \varphi_k \leq ||w_k||_{L^2} ||\varphi_k||_{L^2}. \tag{2.59}$$

Since $\int_\Omega \varphi_k dx = 0$, by *Poincaré*'s inequality, $||\varphi_k||_{L^2} \leq c||\nabla \varphi_k||_{L^2}$, therefore (2.58) and (2.59) imply

$$||\nabla \varphi_k||_{L^2} \leq c||w_k||_{L^2} \tag{2.60}$$

and

$$\int_\Omega \nabla(|w_k|^{p-1} w_k) \nabla \varphi_k dx = \int_\Omega |w_k|^{p+1} dx. \tag{2.61}$$

Since

$$\nabla(|w_k|^{p-1}w_k) = p|w_k|^{p-1}\nabla w_k, \tag{2.62}$$

we have

$$|\nabla(|w_k|^{p-1}w_k)| = p|w_k|^{p-1}|\nabla w_k| = |\nabla|w_k|^p|. \tag{2.63}$$

Hence, from (2.61), we have

$$\begin{aligned} \int_\Omega |w_k|^{p+1}dx &= \int_\Omega \nabla(|w_k|^{p-1}w_k)\nabla\varphi_k dx \\ &\le \int_\Omega |\nabla|w_k|^p||\nabla\varphi_k|dx \\ &\le ||\nabla|w_k|^p||_{L^2}||\nabla\varphi_k||_{L^2} \\ &\to 0 \end{aligned} \tag{2.64}$$

as $k \to \infty$, by (2.52) and (2.60).

Hence, along a subsequence,

$$|w_k|^{p+1} \to 0 \ a.e \ in \ \Omega,$$

i.e,

$$|w_k| \to 0 \ a.e \ in \ \Omega. \tag{2.65}$$

This contradicts (2.57).

□

Remark 2.13. After this was complete, we became aware of a similar result by Alikakos and Rostamian in [4] but we include our result for completeness.

The next step is to establish the existence of an absorbing set in L^{q+1} for all $q > 1$. This is done by first writing the equation in terms of $v = u - u_0$, multiplying that equation by $v|v|^{q-1}$, and integrating. Then one uses Proposition 2.12 and a uniform Gronwall inequality found in [75] to obtain the following:

Proposition 2.14. *Let $\alpha_0 < (\frac{c_1}{c_2})^{\frac{1}{2r}}$, where c_1 and c_2 are the constants in assumption (A_2), and let $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 dx$. If $u(x)$ is a solution of (2.1)-(2.3), and $|\bar{u}_0| \le \alpha_0$, then for any $q > 1$, we have*

$$\int_\Omega |u - \bar{u}_0|^{q+1}dx < C_1 + (\frac{C_2 rt}{q+1})^{-\frac{q+1}{2r}} \tag{2.66}$$

where C_1 depends on α_0 and q, and C_2 depends on q. Consequently, one has for any solution of (2.1)-(2.3) with $\frac{1}{|\Omega|}|\int u_0 dx| = |\bar{u}_0| \le \alpha_0$, there exists a time $t_0(\alpha_0, q) \ge 0$ such that

$$||u||_{q+1} < \mu, \ for \ all \ t > t_0(\alpha_0, q), \tag{2.67}$$

where

$$\mu > (\frac{C_3(q)}{C_4(q,\alpha_0)})^{\frac{1}{q+2r+1}} + \alpha_0|\Omega|^{\frac{1}{q+1}}.$$

We have shown that in L^p there exists a "local absorbing set" in the sense that if $|\int u_0|$ is not too large, the solution enters a fixed bounded set in the affine space $\bar{u}_0 + L^p$ in finite time (note that $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0$ is conserved by the evolution). Now we consider the long term behavior of the solution in the H^1 norm. In this case, we do not need any restriction on $|\int u_0|$.

Note that (A_2) implies

$$f(u)u \geq c_5|u|^{2r+2} - c_6 \text{ for some constants } c_5 \text{ and } c_6.$$

We make additional assumptions on the nonlinearity,

(A_4) $|f(u)| \leq c_7|u|^{2r+1} + c_8$,
(A_5) $F(u) = \int_0^u f(s)ds \leq c_9|u|^{2r+2} + c_{10}$, and $c_5 > c_9$.

Remark 2.15. (A_2), (A_4), and (A_5) hold for $f(u) = c|u|^{2r}u+$ lower terms.

Denote $\bar{\psi} = \frac{1}{|\Omega|}\int_\Omega \psi dx$, write $\varphi = \psi - \bar{\psi}$.

For $\varphi \in L^2(\Omega)$, satisfying $\bar{\varphi} = 0$, we consider the following equation:

$$\begin{cases} -\triangle\theta = \varphi \\ \frac{\partial\theta}{\partial n}|_{\partial\Omega} = 0 \\ \int_\Omega \theta = 0 \end{cases} \tag{2.68}$$

The equation (2.68) has a unique solution $\theta := (-\triangle_0)^{-1}(\varphi)$. Denote $||\varphi||_{-1} = (\int_\Omega (-\triangle_0)^{-1}(\varphi)\varphi dx)^{\frac{1}{2}}$. This is a continuous norm on $L^2(\Omega)$.

Since $\bar{u} = \bar{u}_0$ is constant, we may write the equation as

$$\frac{\partial(u-\bar{u})}{\partial t} = \triangle K(u), \tag{2.69}$$

where $K(u) = \int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)$. Applying the operator $(-\triangle_0)^{-1}$ to both sides of equation (2.69), we obtain

$$\frac{d(-\triangle_0)^{-1}(u-\bar{u})}{dt} + K(u) = 0. \tag{2.70}$$

Taking the scalar product with $u - \bar{u}$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{d}{dt}||u-\bar{u}||_{-1}^2 + (K(u), u-\bar{u}) = 0. \tag{2.71}$$

From condition (A_2)–(A_5), we have

$$(K(u), u - \bar{u}) \geq \frac{1}{2}\int\int J(x-y)(u(x)-u(y))^2 dxdy + c_5 \int |u|^{2r+2}dx - \epsilon \int |u|^{2r+2}dx - c(\bar{u}, \epsilon) \tag{2.72}$$

for any $\epsilon > 0$. Choosing $\epsilon = c_5 - c_9$, we have

$$\begin{aligned}(K(u), u - \bar{u}) &\geq \frac{1}{2}\int\int J(x-y)(u(x)-u(y))^2 dxdy + c_9 \int |u|^{2r+2}dx - c(\bar{u}) \\ &\geq E(u) - c(\bar{u}) \\ &= E(u) - c(\bar{u}_0).\end{aligned} \tag{2.73}$$

Also from (2.73), we have

$$(K(u), u - \bar{u}) \geq c\int |u|^{2r+2}dx - c(\bar{u}_0) \tag{2.74}$$

for some positive constants c and $c(\bar{u}_0)$.

Since $||.||_{-1}$ is a continuous norm on $L^2(\Omega)$, we have

$$||u - \bar{u}||_{-1} \leq C||u - \bar{u}||_2. \tag{2.75}$$

Therefore,

$$\begin{aligned}||u - \bar{u}_0||_{-1} &\leq C||u - \bar{u}_0||_2 \\ &\leq C||u||_{2r+2} + C(\bar{u}_0)\end{aligned} \tag{2.76}$$

for some positive constants C and $C(\bar{u}_0)$. From (2.71), (2.74), and (2.76), it follows that

$$\frac{d}{dt}||u - \bar{u}_0||_{-1}^2 + C||u - \bar{u}_0||_{-1}^{2r+2} \leq C(\bar{u}_0). \tag{2.77}$$

By the uniform Gronwall inequality mentioned above, we obtain

$$||u - \bar{u}_0||_{-1}^2 \leq (\frac{C(\bar{u}_0)}{C})^{\frac{1}{r+1}} + (C(r)t)^{\frac{-1}{r}}. \tag{2.78}$$

Thus, we have proved:

Theorem 2.16. *There exists $M(\bar{u}_0)$ such that for any $\rho > M(\bar{u}_0)^{\frac{1}{2r+2}}$, there exists a time t_0 such that*

$$||u - \bar{u}_0||_{-1} \leq \rho, \ \ \forall\, t \geq t_0. \tag{2.79}$$

From (2.71) and (2.72), we also obtain

$$\frac{1}{2}\frac{d}{dt}||u-\bar{u}_0||_{-1}^2 + E(u) \le c(\bar{u}_0). \tag{2.80}$$

Integrating from t to $t+1$, then (2.79) implies

$$\int_t^{t+1} E(u(s))ds \le c^*(\bar{u}_0) \equiv c(\bar{u}_0) + \frac{\rho^2}{2} \tag{2.81}$$

for $t \ge t_0$. Since $E(u(t))$ is decreasing, (2.81) implies

$$E(u(t)) \le c^*(\bar{u}_0) \tag{2.82}$$

for $t \ge t_0 + 1$.

Since, from (2.6),

$$\begin{aligned} E(u(t)) &\ge \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2dxdy + \int F(u)dx \\ &\ge c_3\int |u|^{2r+2} - c_4, \end{aligned} \tag{2.83}$$

inequalities (2.82) and (2.83) yield

$$\int |u|^{2r+2} \le c_*(\bar{u}_0) \tag{2.84}$$

for $t \ge t_0$.

Corollary 2.17. *There exists $c_*(\bar{u}_0) > M(\bar{u}_0)^{\frac{1}{2r+2}}$ such that for any $\rho > c_*(\bar{u}_0)$, there exists a time t_0^* such that*

$$\int |u|^{r+1} \le c_*(\bar{u}_0) \ for\ t \ge t_0^*. \tag{2.85}$$

Next we estimate $||\nabla u||_2$.

Denote $h(x,u) = a(x)u + f(u)$, multiplying (2.1) by $h(x,u)$ and integrating over Ω, we have

$$\int h(x,u)u_t + \int |\nabla h(x,u)|^2 = \int \nabla J * u \cdot \nabla h(x,u). \tag{2.86}$$

Since

$$h(x,u)u_t = (a(x)u + f(u))u_t = \frac{\partial}{\partial t}[\frac{1}{2}a(x)u^2 + F(u)], \tag{2.87}$$

and

$$\int \nabla J * u \cdot \nabla h(x,u) \le c||u||_2^2 + \frac{1}{2}||\nabla h(x,u)||_2^2, \tag{2.88}$$

equation (2.86) yields

$$\frac{d}{dt}\int [\frac{1}{2}a(x)u^2 + F(u)] + \frac{1}{2}\int |\nabla h(x,u)|^2 \le c||u||_2^2. \tag{2.89}$$

Integrate (2.89) from t to $t+1$, and use assumption (A_2) and Corollary 2.17, to obtain

$$\int_t^{t+1} \int |\nabla h(x,u)|^2 \leq c \tag{2.90}$$

for some constant c and all $t \geq t_0^*$.

Multiply (2.1) by $h(x,u)_t$ and integrate on Ω, to obtain

$$\int h(x,u)_t u_t + \int \nabla h(x,u) \cdot \nabla h(x,u)_t = \int \nabla J * u \cdot \nabla h(x,u)_t. \tag{2.91}$$

Since

(2.92)

$$h(x,u)_t u_t = a(x)u_t^2 + f'(u)u_t^2 \geq c_1 u_t^2,$$

$$\int \nabla h(x,u) \cdot \nabla h(x,u)_t = \frac{1}{2}\frac{d}{dt}\int |\nabla h(x,u)|^2,$$

and

$$\int \nabla J * u \cdot \nabla h(x,u)_t = \frac{d}{dt}\int \nabla J * u \cdot \nabla h(x,u) - \int \nabla J * u_t \cdot \nabla h(x,u),$$

we have

$$\begin{aligned} c_1 \int |u_t|^2 &+ \frac{1}{2}\frac{d}{dt}\int |\nabla h(x,u)|^2 \\ &\leq \frac{d}{dt}\int \nabla J * u \cdot \nabla h(x,u) - \int \nabla J * u_t \cdot \nabla h(x,u). \end{aligned} \tag{2.93}$$

Estimate (2.93) with the Cauchy-Schwartz, and Young's inequalities imply

$$\frac{d}{dt}\int |\nabla h(x,u)|^2 \leq \frac{d}{dt}\int 2\nabla J * u \cdot \nabla h(x,u) + \gamma \int |\nabla h(x,u)|^2 \tag{2.94}$$

for some constant $\gamma > 0$.

For $t < s < t+1$, multiplying (2.94) by $e^{\gamma(t-s)}$, we have

$$\frac{d}{ds}[e^{\gamma(t-s)}\int |\nabla h(x,u)|^2] \leq e^{\gamma(t-s)}\frac{d}{ds}\int 2\nabla J * u \cdot \nabla h(x,u). \tag{2.95}$$

Integrating (2.95) between s and $t+1$, we obtain

$$\begin{aligned} e^{-\gamma}\int_\Omega |\nabla h(x,u(x,t+1))|^2 &- e^{\gamma(t-s)}\int |\nabla h(x,u(x,s))|^2 \\ \leq \int_s^{t+1} e^{\gamma(t-\mu)}\frac{d}{d\mu}&\int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu))dxd\mu. \end{aligned} \tag{2.96}$$

Since

$$\begin{aligned}
&\int_s^{t+1} e^{\gamma(t-\mu)} \frac{d}{d\mu} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx d\mu \\
&= e^{\gamma(t-\mu)} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx |_s^{t+1} \\
&\quad - \int_s^{t+1} (-\gamma) e^{\gamma(t-\mu)} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx d\mu \\
&= I_1 + I_2.
\end{aligned} \tag{2.97}$$

These may be individually estimated yielding

(2.98)

$$\begin{aligned}
&e^{-\gamma} \int_\Omega |\nabla h(x,u(x,t+1))|^2 - e^{\gamma(t-s)} \int_\Omega |\nabla h(x,u(x,s))|^2 \leq \\
&\frac{e^{-\gamma}}{2} \int_\Omega |\nabla h(x,u(x,t+1))|^2 + C \int_\Omega |u(x,t+1)|^2 + \int |\nabla h(x,u(x,s))|^2 \\
&+ C \int_\Omega |u(x,s)|^2 + C \int_s^{t+1} [\int_\Omega |\nabla h(x,u(x,\mu))|^2 + \int_\Omega |u(x,\mu)|^2] d\mu.
\end{aligned}$$

Therefore,

(2.99)

$$\begin{aligned}
&\frac{e^{-\gamma}}{2} \int_\Omega |\nabla h(x,u(x,t+1))|^2 \\
&\leq e^{\gamma(t-s)} \int_\Omega |\nabla h(x,u(x,s))|^2 + C \int_\Omega |u(x,t+1)|^2 + \int_\Omega |\nabla h(x,u(x,s))|^2 \\
&+ C \int_\Omega |u(x,s)|^2 + C \int_s^{t+1} [\int_\Omega |\nabla h(x,u(x,\mu))|^2 + \int_\Omega |u(x,\mu)|^2] d\mu.
\end{aligned}$$

Integrating (2.99) from t to $t+1$ with respect to s, we have

$$\begin{aligned}
&\frac{e^{-\gamma}}{2} \int_\Omega |\nabla h(x,u(x,t+1))|^2 dx \\
&\leq \int_t^{t+1} \int_\Omega |\nabla h(x,u(x,s))|^2 dx ds + C \int_\Omega |u(x,t+1)|^2 dx \\
&+ \int_t^{t+1} \int_\Omega |\nabla h(x,u(x,s))|^2 dx ds + C \int_t^{t+1} \int_\Omega |u(x,s)|^2 dx ds \\
&+ C \int_t^{t+1} (\mu - t) [\int_\Omega |\nabla h(x,u(x,\mu))|^2 dx + \int_\Omega |u(x,\mu)|^2 dx] d\mu.
\end{aligned} \tag{2.100}$$

By (2.85) and (2.90), estimate (2.100) yields

$$\int_\Omega |\nabla h(x,u(x,t+1))|^2 dx \leq C(\bar{u}_0) \tag{2.101}$$

for $t \geq t_0(\bar{u}_0)$ and some $C(\bar{u}_0) > 0$.

Since

(2.102)
$$\nabla h(x,u(x,t+1)) = (a(x)+f'(u(t+1)))\nabla u(x,t+1) - u(x,t+1)\nabla a(x),$$

we have

(2.103)
$$\begin{aligned}\int_\Omega |\nabla h(x,u(x,t+1))|^2 &\geq \frac{1}{2}\int_\Omega |(a(x)+f'(u(t+1)))|^2|\nabla u(x,t+1)|^2 \\ &\quad - \int_\Omega |u(x,t+1)\nabla a(x)|^2 \\ &\geq \int_\Omega \frac{1}{2}c_1^2|\nabla u(x,t+1)|^2 - D(\bar{u}_0)\end{aligned}$$

for $t \geq t_0(\bar{u}_0)$ and some constant $D(\bar{u}_0)$.

Estimates (2.101) and (2.103) imply

$$\int_\Omega |\nabla u(x,t+1)|^2 \leq G(\bar{u}_0), \tag{2.104}$$

for $t \geq t_0^*(\bar{u}_0)$ and $G(\bar{u}_0) > 0$. Thus, we have

Theorem 2.18. *There exists a time* $t_0^*(\bar{u}_0)$ *such that*

$$||u||_{H^1} \leq c(\bar{u}_0) \ for\ t \geq t_0^*(\bar{u}_0). \tag{2.105}$$

Remark 2.19. [75] gives a similar result for the Cahn-Hilliard equation.

This boundedness gives weak convergence of subsequences as $t_n \to \infty$ but more is true, as can be demonstrated by calculations similar to the foregoing:

Theorem 2.20. *If* u *is a solution of (2.1)-(2.3), and* $Q(u) = (\int_\Omega J(x-y)dy)u(x) - J * u(x) + f(u(x)),$ *then there exist a sequence* $\{t_k\}$ *and* u^* *such that*

$$\begin{aligned} u(t_k) &\rightharpoonup u^* \ \ weakly\ \ in\ H^1 \\ Q(u(t_k)) &\rightharpoonup Q(u^*) \ \ weakly\ \ in\ H^1 \end{aligned} \tag{2.106}$$

and $Q(u^*)$ *is a constant, i.e.* u^* *is a steady state solution of (2.1)-(2.3).*

We may also use the above techniques for the following integrodifferential equation that may be derived from interacting particle systems with Kawasaki dynamics

$$\begin{cases} \frac{\partial u}{\partial t} = \triangle(u - tanh\beta J * u) & \text{in} \quad \Omega \\ \frac{\partial(u - tanh\beta J*u)}{\partial n} = 0 & \text{on} \quad \partial\Omega \\ u(x,0) = u_0(x) \end{cases} \tag{2.107}$$

where β is a constant, J is a smooth function.

Wellposedness and regularity of solutions is established along the lines used for (2.1)-(2.3) with the usual smoothness assumptions on J, f, Ω, and the initial data. Note that the average of u is constant in time and one can show that there is an absorbing set in every constant mass affine subspace of H^1.

Returning to the nonlocal Cahn-Hilliard equation (2.1), we may also append the homogeneous Dirichlet boundary condition

$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega. \tag{2.108}$$

While we no longer have conservation of the integral, this boundary condition is strongly dissipative and so we expect results similar to those above. In particular with condition

$(A_2)_D$ There exists $c_1 > 0$ such that $a(x,u) \equiv a(x) + f'(u) \geq c_1$,

one can prove

Proposition 2.21. *Assume* $(A_1), (A_2)_D$, *and* (A_3). *If* $u(x,t) \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ *is a solution of (2.1),(2.108), with initial data* u_0 *then*

$$\max_{\bar{Q}_T} |u| \leq C(\Omega, T, u_0) \tag{2.109}$$

for some positive constant $C(\Omega, T, u_0)$.

This is proved by letting $u(x,t) = v(x,t)e^{\sigma t}$ for appropriate choice of σ, multiplying the v-equation by v to obtain an equation for v^2, and applying a maximum principle. A result similar to Theorem 2.4 gives gradient and Hölder-$(1+\alpha)$ bounds on the solution for small $\alpha > 0$. Then it is straightforward to apply the Schauder Fixed Point Theorem to establish the existence of a classical solution.

Under the assumption $(A_2)_D$, equation (2.1) is a nondegenerate parabolic equation. We may also consider the degenerate case. Consider the following equation with $u_0 \in L^\infty(\Omega)$

$$\begin{cases} \frac{\partial u}{\partial t} = \triangle(h(x,u)) - \int_\Omega u(y) \triangle J(x-y)dy & \text{in} \quad Q_T \\ u = 0 & \text{on} \quad S_T \\ u(x,0) = u_0(x), \end{cases} \tag{2.110}$$

where

$$h(x,u) = a(x)u(x) + f(u).$$

Instead of nondegeneracy condition $(A_2)_D$, we assume:

(B_1) For every fixed x, $h(x,0) = 0$, and $\frac{\partial h(x,u)}{\partial u} \geq d_1 |u|^{r_1}$ for some positive constants r_1 and d_1.

Definition 2.22. A generalized solution of (2.110) is a function $u \in C([0,T] : L^1(\Omega)) \cap L^\infty(Q_T)$ such that

(2.111)
$$\int_\Omega u(x,t)\psi(x,t)dx - \int\int_{Q_t} u(x,t)\psi_s(x,s)dxds = \int\int_{Q_t} h(x,u)\triangle\psi(x,s)dxds - \int\int_{Q_t} (\triangle J * u(\cdot,s))\psi(x,s)dxds + \int_\Omega u(x,0)\psi(x,0)dx$$

for all $\psi \in C^{2,1}(\bar{Q}_T)$ such that $\psi(x,t) = 0$ for $x \in \partial\Omega$ and $0 \le t \le T$, and

$$u(x,0) = u_0(x). \tag{2.112}$$

We first prove the uniqueness.

Proposition 2.23. *Let u_1, u_2 be two solutions of equation (2.110) with initial data u_{10}, $u_{20} \in L^\infty(\Omega)$, then*

$$||u_1(\tau) - u_2(\tau)||_{L^1(\Omega)} \le C(T)||u_{10} - u_{20}||_{L^1(\Omega)}$$

for each $\tau \in (0,T)$, and some constant $C(T)$.

Proof. For any $\tau \in (0,T)$, and $\psi \in C^{2,1}(\bar{Q}_\tau)$ with $\psi|_{\partial\Omega} = 0$ for $0 < t < \tau$, after multiplying (2.110) by ψ and integrating over $\Omega \times (0,\tau)$, we have

(2.113)
$$\int_\Omega u_i(x,\tau)\psi(x,\tau)dx = \int_\Omega u_i(x,0)\psi(x,0)dx + \int_0^\tau \int_\Omega (u_i\psi_t + h(x,u_i)\triangle\psi)dxdt + \int_0^\tau \int_\Omega (\triangle J * u_i)\psi dxdt.$$

Setting $z = u_1 - u_2$ and $z_0 = u_{10} - u_{20}$, equation (2.113) gives

(2.114)
$$\int_\Omega z(x,\tau)\psi(x,\tau)dx = \int_\Omega z_0(x)\psi(x,0)dx + \int_0^\tau \int_\Omega z(\psi_t + b(x,t)\triangle\psi)dxdt + \int_0^\tau \int_\Omega (\triangle J * z)\psi dxdt,$$

where

$$b(x,t) = \begin{cases} \frac{h(x,u_1)-h(x,u_2)}{u_1-u_2} & \text{for} \quad u_1 \ne u_2, \\ h_u(x,u_1) & \text{for} \quad u_1 = u_2. \end{cases}$$

Follow the idea in [6], we consider problem:

$$
(2.115) \qquad \begin{cases} \frac{\partial \psi}{\partial t} = -b\triangle\psi + \nu\psi & \text{in} \quad \Omega,\ 0<t<\tau, \\ \psi = 0 & \text{on} \quad \partial\Omega,\ 0<t<\tau, \\ \psi(x,\tau) = g(x), \end{cases}
$$

where $g(x) \in C_0^\infty(\Omega)$, $0 \le g \le 1$, and $\nu > 0$ is constant.

Since b just belongs to $L^\infty(Q_T)$ and may be equal to zero, we perturb to get a nondegenerate equation, by setting $b_n = \rho_n * b + \frac{1}{n}$, where ρ_n is a mollifier in $\mathbb{R}^n$, and $\int_0^\tau \int_\Omega (\rho_n * b - b)^2 dxdt \le \frac{1}{n^2}$. Consider

$$
(2.116) \qquad \begin{cases} \frac{\partial \psi}{\partial t} = -b_n\triangle\psi + \nu\psi & \text{in} \quad \Omega,\ 0<t<\tau, \\ \psi = 0 & \text{on} \quad \partial\Omega,\ 0<t<\tau, \\ \psi(x,\tau) = g(x). \end{cases}
$$

Since $b_n \ge \frac{1}{n}$, the equation is a nondegenerate parabolic equation, and so there exists a solution $\psi_n \in C^{2,1}(\bar{Q}_\tau)$.

The following, whose proof we omit, is easily established.

Lemma 2.24. *The solution of (2.116) has the following properties*

$$
\begin{aligned}
&(i)\ \ 0 \le \psi_n \le e^{\nu(t-\tau)}, \\
&(ii)\ \ \int_0^\tau \int_\Omega b_n |\triangle(\psi_n)|^2 dxdt \le C, \\
&(iii)\ \ \sup_{0\le t\le\tau} \int_\Omega |\nabla\psi_n|^2 dx \le C,
\end{aligned}
$$

where the constant C depends only on g.

Replacing ψ by ψ_n in (2.114), and using (2.116) we obtain

(2.117)
$$
\begin{aligned}
&\int_\Omega z(x,\tau)g(x)dx - \int_0^\tau \int_\Omega z(b-b_n)\triangle\psi_n dxdt \\
&\qquad = \int_\Omega z(x,0)\psi_n(0)dx + \int\int_{Q_\tau} (\triangle J * z + \nu z)\psi_n dxdt.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_0^\tau \int_\Omega z(b-b_n)\triangle\psi_n dxdt \\
&\le C\Big(\int_0^\tau \int_\Omega \frac{(b-b_n)^2}{b_n} dxdt\Big)^{\frac{1}{2}} \Big(\int_0^\tau \int_\Omega b_n |\triangle\psi_n|^2 dxdt\Big)^{\frac{1}{2}} \\
&\le \frac{C}{\sqrt{n}} \to 0,
\end{aligned}
$$

equation (2.117) implies

$$\int_\Omega z(x,\tau)g(x)dx \tag{2.118}$$
$$\le \int_\Omega |z(x,0)|e^{\nu(t-\tau)}dx + \int\int_{Q_\tau} |\triangle J * z + \nu z|e^{\nu(t-\tau)}dxdt.$$

Letting $\nu \to 0$ and $g(x) \to signz^+(x,\tau)$ in (2.118), we have

$$\int_\Omega (u_1 - u_2)^+ dx \le \int_\Omega |u_{10} - u_{20}|dx + \int\int_{Q_\tau} |\triangle J * z|dxdt. \tag{2.119}$$

Interchanging u_1 and u_2 yields

$$\int_\Omega |u_2 - u_1|dx \le \int_\Omega |u_{20} - u_{10}|dx + C\int\int_{Q_\tau} |u_2 - u_1|dxdt. \tag{2.120}$$

(2.120) and Gronwall's inequality imply the conclusion.

□

Remark 2.25. Since every classical solution is also a weak solution, this also proves the uniqueness and continuous dependence on initial values for classical solutions.

To prove the existence of a solution to (2.111), we consider the regularized problem and take $u_0 \in C^{2+\alpha}(\bar{\Omega})$ for some $\alpha > 0$, with $u_0|_{\partial\Omega} = 0$:

$$\begin{cases} \frac{\partial u}{\partial t} = \triangle(h^\epsilon(x,u)) - \int_\Omega \triangle J(x-y)u(y)dy & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad S_T, \\ u(x,0) = u_0(x), \end{cases} \tag{2.121}$$

where

$$h^\epsilon(x,u) = a(x)u(x) + f(u) + \epsilon u.$$

We have shown that there exists a classical solution $u_\epsilon(x,t) \in C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q}_T)$. It is easy to show that these solutions are uniformly bounded on $\bar{Q}_T$.

Using the growth conditions and Arzela-Ascoli's lemma, one can then prove

Theorem 2.26. *For any $T > 0$ and $u_0 \in L^\infty(\Omega)$, if conditions (A_1), (B_1), and (A_3) are satisfied, then there exists a unique function $u \in C([0,T], L^1(\Omega)) \cap L^\infty(Q_T)$ which satisfies equation (2.111).*

Results concerning the long-term behavior of solutions to the Dirichlet problem follow from similar ideas introduced for the case of no-flus boundary conditions but this time a nonlinear version of the *Poincaré* inequality is not used.

In order to prove the existence of an absorbing set, instead of $(A_2)_D$, we assume the original (A_2) and

(A_4) There exist positive constants c_3 and c_4 such that $a(x,u) \leq c_3|u|^r + c_4$.

First one establishes L^p bounds for solutions by using a Gronwall inequality after multiplying the equation by a power of u and integrating, performing some tedious manipulations.

Then one proves

Proposition 2.27. *If $u_0 \in L^\infty(\Omega)$, then*

$$\sup_{t\geq 0} ||u||_\infty \leq C(u_0). \tag{2.122}$$

Also, gradient bounds may be obtained using the above and some messy calculations:

Theorem 2.28. *Assume that u is a solution of (2.1), (2.108) and conditions (A_1) – (A_4) are satisfied. There exists $t_0 > 0$ such that if $t \geq t_0$ then*

$$\sup_{t\geq t_0} ||\nabla u||_2 < C, \tag{2.123}$$

where constant C does not depend on initial data.

If we restrict our attention to one space dimension where better embedding theorems are in force, one can then prove:

Theorem 2.29. *For $n = 1$, if conditions (A_1) – (A_4) are satisfied, then the semigroup associated with (2.1) with Dirichlet boundary conditions possesses an attractor $\mathcal{A} \subset H^1(\Omega) \cap X$ which is maximal and compact.*

3. Nonlocal Phase-Field System

We now turn to the system where the temperature evolves and the order parameter represents local solidification, partially driven by temperature and phase change in turn producing or absorbing heat energy, thus driving temperature. The following presents some results reported in [19]. As outlined above, this system has the form:

$$u_t = J * u - u \int_\Omega J(x-y)dy - f(u) + l\theta, \tag{3.1}$$

$$(\theta + lu)_t = \Delta\theta, \tag{3.2}$$

which is complemented by the initial and boundary conditions

$$u(0,x) = u_0(x),\ \theta(0,x) = \theta_0(x), \tag{3.3}$$

$$\frac{\partial \theta}{\partial n}|_{\partial\Omega} = 0, \tag{3.4}$$

where $T > 0$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. We are interested in the well posedness of this initial and boundary value problem.

In order to prove the existence, we make the following assumptions

(P_1) $M \equiv \sup \int_\Omega |J(x-y)|dy < \infty$ and $f \in C(\mathbb{R})$.

(P_2) There exist $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ and $r > 2$ such that $f(u)u \geq c_1|u|^r - c_2|u|$, and $|f(u)| \leq c_3|u|^{r-1} + c_4$.

Note that (P_2) implies

$$F(u) = \int_0^u f(s)ds \geq c_5|u|^r - c_6|u| \tag{3.5}$$

for some positive constants c_5 and c_6.

We prove the existence of a solution to (3.1)-(3.4) by the method of successive approximation.

Define $\theta^{(0)}(t,x) := \theta_0(x)$ and for $k \geq 1$ $(u^{(k)}, \theta^{(k)})$ iteratively to be solutions to the system

$$u_t^{(k)} = \int_\Omega J(x-y)u^{(k)}(y)dy - \int_\Omega J(x-y)dyu^{(k)}(x) - f(u^{(k)}) + l\theta^{(k-1)}, \tag{3.6}$$

$$\theta_t^{(k)} - \triangle\theta^{(k)} + \theta^{(k)} = -lu_t^{(k)} + \theta^{(k-1)} \tag{3.7}$$

in $(0,T) \times \Omega$, with initial and boundary conditions

$$u^{(k)}(0,x) = u_0(x), \;\; \theta^{(k)}(0,x) = \theta_0(x), \tag{3.8}$$

$$\frac{\partial\theta^{(k)}}{\partial n}|_{\partial\Omega} = 0. \tag{3.9}$$

Lemma 3.1. *With $k = 1$, for any $T > 0$, if $u_0 \in L^\infty(\Omega)$, and $\theta_0 \in H^1 \cap L^\infty(\Omega)$, then there exists a unique solution (u, θ) to system (3.6) -(3.9). Furthermore, $u^{(1)}$, $u_t^{(1)} \in L^\infty((0,T), L^\infty(\Omega))$ and $\theta^{(1)} \in L^\infty((0,T), L^\infty(\Omega)) \cap L^2((0,T), H^2(\Omega))$.*

Proof. Since the right hand side of equation (3.1) is locally Lipschitz continuous in $L^\infty((0,T), L^\infty(\Omega))$, local existence follows from standard ODE theory. In order to prove the global existence, we prove global boundedness of the solutions. For any $p > 1$, multiplying equation (3.1) by $|u^{(1)}|^{p-1}u$ and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\int |u^{(1)}|^{p+1}dx + \int f(u^{(1)})|u^{(1)}|^{p-1}udx \\ &= \int\int J(x-y)u^{(1)}(y)|u^{(1)}|^{p-1}u^{(1)}dxdy \\ &- \int\int J(x-y)u^{(1)}(x)|u^{(1)}|^{p-1}u^{(1)}dxdy + l\int \theta^{(0)}|u^{(1)}|^{p-1}udx. \end{aligned} \tag{3.10}$$

Using Holder's and Young's inequalities and conditions (P_1) and (P_2), we have

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\int |u^{(1)}|^{p+1}dx + C\int |u^{(1)}|^{p+r-1}udx \\ &\leq C(p){C_1}^{p+1}, \end{aligned} \tag{3.11}$$

where C_1 is a constant independent of p and $lim_{p\to\infty}C(p)^{\frac{1}{p+1}} \leq C_2$ with C_2 independent of p.

Using the uniform Gronwall inequality and (3.11), we have

$$||u^{(1)}||_{p+1}^{p+1} \leq (C(p)C_1^{p+1})^{\frac{p+1}{p+r-1}} + (C(r-2)t)^{\frac{-(p+1)}{r-2}}. \tag{3.12}$$

Therefore,

$$||u^{(1)}||_{p+1} \leq C(p)^{\frac{1}{p+1}}(C_1)^{\frac{p+1}{p+r-1}} + (C(r-2)t)^{\frac{-1}{r-2}}. \tag{3.13}$$

Letting $p \to \infty$, we have

$$||u^{(1)}||_\infty \leq C. \tag{3.14}$$

for some constant C.

Also from condition (P_2) and equation (3.6), we have

$$||u_t^{(1)}||_\infty \leq C. \tag{3.15}$$

Since equation (3.7) is a linear parabolic equation, by inequality (3.15) and standard parabolic theory, we have $\theta^{(1)} \in L^\infty((0,T), L^\infty(\Omega)) \cap L^2((0,T), H^2(\Omega))$.

□

By induction, there exist unique solution $(u^{(k)}, \theta^{(k)})$ of system (3.6)-(3.8). Furthermore, $u^{(k)}$, $u_t^{(k)} \in L^\infty((0,T), L^\infty(\Omega))$ and $\theta^{(k)} \in L^\infty((0,T), L^\infty(\Omega)) \cap L^2((0,T), H^2(\Omega))$ for every k. Now we prove that there exists a uniform bound for $u^{(k)}$, $u_t^{(k)}$ and $\theta^{(k)}$.

Multiplying equation (3.7) by $|\theta^{(k)}|^{p-1}\theta^{(k)}(x)$ for $p > \frac{n}{2}$, and integrating over Ω, we have

(3.16)

$$\begin{aligned}
&\int |\theta^{(k)}|^{p-1}\theta^{(k)}\theta_t^{(k)}dx + \int \nabla(|\theta^{(k)}|^{p-1}\theta^{(k)}) \cdot \nabla\theta^{(k)}dx + \int |\theta^{(k)}|^{p+1}dx \\
&= -l\int\int J(x-y)u^{(k)}(y)|\theta^{(k)}|^{p-1}\theta^{(k)}dydx + l\int f(u^{(k)})|\theta^{(k)}|^{p-1}\theta^{(k)}dx \\
&+ l\int\int J(x-y)u^{(k)}(x)|\theta^{(k)}|^{p-1}\theta^{(k)}dydx + (1-l^2)\int |\theta^{(k)}|^{p-1}\theta^{(k)}\theta^{(k-1)}dx.
\end{aligned}$$

Since

$$|\nabla|\theta|^{\frac{p+1}{2}}|^2 = \frac{(p+1)^2}{4}|\theta|^{p-1}|\nabla\theta|^2 = \frac{(p+1)^2}{4p}\nabla(|\theta|^{p-1}\theta)\cdot\nabla\theta, \tag{3.17}$$

using Holder's and Young's inequalities, we obtain

(3.18)

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |\theta^{(k)}|^{p+1}dx + \frac{4p}{(p+1)^2}\int |\nabla|\theta^{(k)}|^{\frac{p+1}{2}}|^2dx + \frac{1}{2}\int |\theta^{(k)}|^{p+1}dx \\
&\leq c_1(l,p)\int |u^{(k)}|^{p+1}dx + c_2(l,p)\int |\theta^{(k-1)}|^{p+1}dx + \int |f(u^{(k)})|^{p+1}dx
\end{aligned}$$

for some positive constants $c_1(l,p)$ and $c_2(l,p)$ which depend only on p and l.

Multiplying equation (3.6) by $|u^{(k)}|^{(r-1)p-1}u^{(k)}$, and integrating over Ω, we obtain

(3.19)
$$\frac{1}{(r-1)p+1}\frac{d}{dt}\int |u^{(k)}|^{(r-1)p+1}dx + \int f(u^{(k)})|u^{(k)}|^{(r-1)p-1}u^{(k)}dx$$
$$= \int\int J(x-y)u^{(k)}(y)|u^{(k)}|^{(r-1)p-1}u^{(k)}dxdy + l\int \theta^{(k-1)}|u^{(k)}|^{(r-1)p-1}u^{(k)}dx$$
$$-\int\int J(x-y)u^{(k)}(x)|u^{(k)}|^{(r-1)p-1}u^{(k)}dxdy.$$

Condition (A_2) implies

$$f(u)|u|^{(r-1)p-1}u \geq c_1|u|^{(r-1)(p+1)} - c_2|u|^{(r-1)p} \tag{3.20}$$

and

$$|f(u)|^{p+1} \leq c_7|u|^{(r-1)(p+1)} + c_8 \tag{3.21}$$

for some positive constants c_7 and c_8. From equation (3.19), inequality (3.20), Holder's and Young's inequalities, we have

(3.22)
$$\frac{1}{(r-1)p+1}\frac{d}{dt}\int |u^{(k)}|^{(r-1)p+1}dx + \frac{c_1}{2}\int |u^{(k)}|^{(r-1)(p+1)}dx$$
$$\leq c(r,p) + c_1(r,p,l)\int |\theta^{(k-1)}|^{p+1}dx$$

for some positive constants $c(r,p)$ and $c_1(r,p,l)$.

Integrating (3.22) from 0 to t, we obtain

(3.23)
$$\frac{1}{(r-1)p+1}\int |u^{(k)}|^{(r-1)p+1}dx + \frac{c_1}{2}\int_0^t\int |u^{(k)}|^{(r-1)(p+1)}dx$$
$$\leq c(r,p)t + c_1(r,p,l)\int_0^t\int \theta^{(k-1)}|^{p+1}dx + \int |u_0|^{(r-1)p+1}$$
$$\leq c(u_0,T,r,p) + c_1(r,p,l)\int_0^t\int \theta^{(k-1)}|^{p+1}dx$$

for some positive constants $c(u_0,T,r,p)$ and $c_1(r,p,l)$.

Integrating inequality (3.18) from 0 to t, using (3.21) and (3.23), we have

$$\int_\Omega |\theta^{(k)}|^{p+1}dx \leq c(u_0,\theta_0,p,r,l,T)(1 + \int_0^t\int |\theta^{(k-1)}|^{p+1}dxds) \tag{3.24}$$

for some positive constant $c(u_0,\theta_0,p,r,l,T)$ which does not depend on k.

By induction, we have

$$\int_\Omega |\theta^{(k)}|^{p+1}dx \leq ce^{ct} \tag{3.25}$$

for some positive constant c which does not depend on k.

Similarly from inequalities (3.22) and (3.25), we also have

$$\int_\Omega |u^{(k)}|^{p+1} dx \le C, \tag{3.26}$$

and

$$\int_\Omega |f(u^{(k)})|^{p+1} dx \le C \tag{3.27}$$

for some positive constant C which does not depend on k.

Equation (3.6), inequalities (3.25)-(3.27), and Young's inequality imply

$$\int_\Omega |u_t^{(k)}|^{p+1} dx \le C \tag{3.28}$$

for some positive constant C which does not depend on k.

This implies $-lu_t^{(k)} + \theta^{(k-1)} \in L^{p+1}((0,T), L^{p+1}(\Omega))$ and

$$|| - lu_t^{(k)} + \theta^{(k-1)}||_{p+1} \le C \tag{3.29}$$

for some positive constant C which does not depend on k.

Applying standard parabolic estimates to equation (3.7), and using inequality (3.28), we have

$$||\theta^{(k)}||_\infty \le C. \tag{3.30}$$

Multiplying equation (3.7) by θ_t^k, and integrating equation (3.7) over Ω, using Holder and Young's inequalities and (3.30), we have

$$\int_0^T \int_\Omega |\theta_t^{(k)}|^2 dx dt \le C \tag{3.31}$$

for some constant C which does not depend on k.

Equation (3.7), inequalities (3.28), (3.30), and (3.31) yield

$$\int_0^T \int_\Omega |\triangle \theta^{(k)}|^2 dx dt \le C \tag{3.32}$$

for some constant C which does not depend on k.

Since $||\theta^{(k)}||_\infty \le C$, using a similar argument to that in the proof of Lemma 3.1, we have

$$||u^{(k)}||_\infty \le C, \tag{3.33}$$

and

$$||u_t^{(k)}||_\infty \le C \tag{3.34}$$

for some constant C which does not depend on k.

Next we prove the convergence of $\{\theta^{(k)}\}$ in $C([0,T],L^2(\Omega))$. From equation (3.7), we have

(3.35)
$$(\theta^{(k+1)}-\theta^{(k)})_t-\triangle(\theta^{(k+1)}-\theta^{(k)})+(\theta^{(k+1)}-\theta^{(k)})$$
$$=-l(u^{(k+1)}-u^{(k)})_t+(\theta^{(k)}-\theta^{(k-1)}).$$

Multiplying equation (3.35) by $(\theta^{(k+1)}-\theta^{(k)})$, and integrating over Ω, using Holder's and Young's inequalities, we have

(3.36)
$$\frac{1}{2}\frac{d}{dt}\int|\theta^{(k+1)}-\theta^{(k)}|^2dx+\int|\nabla(\theta^{(k+1)}-\theta^{(k)})|^2dx+\frac{1}{2}\int(\theta^{(k+1)}-\theta^{(k)})^2$$
$$\leq l^2\int|u_t^{(k+1)}-u_t^{(k)}|^2dx+\int|\theta^{(k)}-\theta^{(k-1)}|^2dx.$$

Since $||u^{(k)}||_\infty\leq C$, from equation (3.6), and condition (P_2), we have

$$\int|u^{(k+1)}-u^{(k)}|^2dx\leq C(T)\int|\theta^{(k)}-\theta^{(k-1)}|^2dx, \tag{3.37}$$

and

(3.38)
$$\int|f(u^{(k+1)})-f(u^{(k)})|^2dx=\int|f'(\lambda u^{(k+1)}+(1-\lambda)u^{(k)})(u^{(k+1)}-u^{(k)})|^2dx$$
$$\leq C(T)\int|u^{(k+1)}-u^{(k)}|^2dx.$$

Therefore, equation (3.6), and inequalities (3.37)-(3.38) imply

(3.39)
$$\int|u_t^{(k+1)}-u_t^{(k)}|^2dx$$
$$\leq 4\int|\int J(x-y)(u^{(k+1)}-u^{(k)})dy|^2dx+4\int(\int J(x-y)dy)^2(u^{(k+1)}-u^{(k)})^2dx$$
$$+4\int(f(u^{(k+1)})-f(u^{(k)})^2dx+4\int(\theta^{(k)}-\theta^{(k-1)})^2dx$$
$$\leq C_1(T)\int|\theta^{(k)}-\theta^{(k-1)}|^2dx$$

for some positive constant $C_1(T)$ which does not depend on k.

Inequalities (3.36)-(3.39) yield

$$\frac{d}{dt}\int|\theta^{(k+1)}-\theta^{(k)}|^2dx\leq C(T)\int|\theta^{(k)}-\theta^{(k-1)}|^2dx \tag{3.40}$$

for some positive constant $C(T)$ which does not depend on k.

By induction, this implies

$$\int|\theta^{(k+1)}-\theta^{(k)}|^2dx\leq\frac{(ct)^{(k-1)}}{(k-1)!}\int_0^t\int|\theta^1-\theta^0|dxds. \tag{3.41}$$

So $\theta^{(k)}$ is a Cauchy sequence in $C([0,T],L^2(\Omega))$. Therefore, there exists $\theta \in C([0,T],L^2(\Omega))$ such that $\theta^{(k)} \to \theta$ in $C([0,T],L^2(\Omega))$. From (3.30)-(3.32), we have

$$||\theta||_\infty \le C, \tag{3.42}$$

$$\int_0^T \int_\Omega |\triangle\theta|^2 dxdt \le C, \tag{3.43}$$

$$\int_0^T \int_\Omega |\theta_t|^2 dxdt \le C. \tag{3.44}$$

Also from (3.33), (3.37)-(3.39), we have

$$u^{(k)} \to u \ in \ C([0,T],L^2(\Omega)), \tag{3.45}$$

$$u_t^{(k)} \to u_t \ in \ C([0,T],L^2(\Omega)), \tag{3.46}$$

$$f(u^{(k)}) \to f(u) \ in \ C([0,T],L^2(\Omega)). \tag{3.47}$$

Therefore, letting $k \to \infty$ in equation (3.6), we have

$$u_t = \int_\Omega J(x-y)u(y)dy - \int_\Omega J(x-y)dyu(x) - f(u) + l\theta \tag{3.48}$$

for $t > 0$ and a.e. $x \in \Omega$.

Since $u_t^{(k)} \rightharpoonup u_t$, $\theta_t^{(k)} \rightharpoonup \theta_t$, $\triangle\theta^{(k)} \rightharpoonup \triangle\theta$ in $L^2((0,T),L^2(\Omega))$, letting $k \to \infty$ in the weak form of equation (3.7), we have

$$\int_0^T \int_\Omega (lu_t + \theta_t)\xi(t,x)dxdt = \int_0^T \int_\Omega \triangle\theta\xi(t,x)dxdt \tag{3.49}$$

for $\xi(t,x) \in L^2((0,T),L^2(\Omega))$.

Since it is true of $\theta^{(k)}$, we also have

$$\int_0^T \int_\Omega \eta(t)(\triangle\theta\varphi + \nabla\theta \cdot \nabla\varphi)dxdt = 0 \tag{3.50}$$

for any $\varphi \in W^{1,2}(\Omega)$ and $\eta \in L^2(0,T)$. This implies $\frac{\partial\theta}{\partial n} = 0$ a.e on $(0,T) \times \partial\Omega$. Also we have

(3.51)

$$\int_\Omega |\theta(0,x) - \theta_0|^2 dx \le 3(\int_\Omega |\theta(0,x) - \theta(t,x)|^2 dx + \int_\Omega |\theta(t,x) - \theta^{(k)}(t,x)|^2 dx + \int_\Omega |\theta^{(k)}(t,x) - \theta_0|^2 dx).$$

Since $\theta^{(k)}(t,x) \to \theta$ in $C([0,T],L^2(\Omega))$, and since $\theta^{(k)}(t,x)$ and $\theta(t,x)$ are continuous with respect to t in $L^2(\Omega)$, by taking k arbitrarily large we can see that $\theta(0,x) = \theta_0$ a.e. in Ω. Similarly, $u(0,x) = u_0$ a.e. in Ω.

Equations (3.48)-(3.51) imply that u and θ are solutions of system (3.1)-(3.4) in a weak sense.

To prove uniqueness and continuous dependence on initial data, let $\theta_{i0} \in L^\infty(\Omega)\cap W^{1,2}(\Omega)$, $u_{i0} \in L^\infty(\Omega)$, and for $R > 0$, $||\theta_{i0}||_{L^\infty} \le R$, $||u_{i0}||_{L^\infty} \le R$, where $i = 1\,, 2$.

Let u_i and θ_i be solutions corresponding to initial data u_{i0} and θ_{i0}, then we have $||\theta_i||_{L^\infty} \le C(T, R)$, and $||u_i||_{L^\infty} \le C(T, R)$.

Denote $v = u_1 - u_2$, $w = \theta_1 - \theta_2$. We have

(3.52)
$$v_t = \int_\Omega J(x-y)v(y)dy - \int_\Omega J(x-y)dyv(x) - f'(\lambda u_1 + (1-\lambda)u_2)v + lw,$$

(3.53)
$$(w + lv)_t = \triangle w$$

in $(0, T) \times \Omega$, for some $\lambda(x, t) \in [0, 1]$. We also have initial and boundary conditions

$$v(0, x) = v_0(x),\ \ w(0, x) = w_0(x), \tag{3.54}$$

$$\frac{\partial w}{\partial n}|_{\partial\Omega} = 0. \tag{3.55}$$

Multiplying equation (3.52) by v_t, integrating over Ω, multiplying equation (3.52) by v, integrating over Ω, multiplying equation (3.53) by w, integrating over Ω, we have

$$\begin{aligned} \int |v_t|^2 = \int\int J(x-y)v(y)dyv_t dx - \int J(x-y)dyv(x)v_t \\ - \int (f'(\lambda u_1 + (1-\lambda)u_2)vv_t + lwv_t)dx, \end{aligned} \tag{3.56}$$

$$\begin{aligned} \int v_t v = \int\int_\Omega J(x-y)v(y)dyvdx - \int_\Omega J(x-y)dyv^2 \\ - \int (f'(\lambda u_1 + (1-\lambda)u_2)v^2 + lwv)dx, \end{aligned} \tag{3.57}$$

$$\int (w_t w + lv_t w) = -\int |\nabla w|^2 dx, \tag{3.58}$$

Adding equations (3.56)-(3.58) together, using Holder's and Young's inequalities, we have

$$\frac{d}{dt}\int [w^2 + v^2]dx \le C_2(T, R)\int [w^2 + v^2]dx \tag{3.59}$$

for some positive constant $C_2(T, R)$.

Inequality (3.59) and Gronwall's inequality imply the uniqueness and continuous dependence on initial data of the solution of (3.6)-(3.7).

Denote $Q_T = (0, T) \times \Omega$, we have the following theorem:

Theorem 3.2. *If assumptions $(P_1), (P_2)$ are satisfied, $u_0 \in L^\infty(\Omega)$ and $\theta_0 \in L^\infty \cap H^1(\Omega)$, then there exists a unique solution $(u, \theta) \in C([0, T], L^\infty(\Omega))$ to the system (3.6)-(3.9) such that $u_t \in L^\infty(Q_T)$, and u_{tt}, θ_t, $\triangle\theta \in L^2(Q_T)$.*

Results concerning the asymptotic behavior of solutions follow along similar, though somewhat more complicated, lines as for the nonlocal Cahn-Hilliard equation above. Here the results are summarized without proof.

Recall that $I_0 \equiv \int(\theta_0 + lu_0)$ is conserved.

Theorem 3.3. *There exists a constant $C(I_0)$ otherwise independent of initial data such that*

$$||u||_r \leq C(I_0),$$
$$||\theta||_{H^1} \leq C(I_0)$$

for $t \geq t_0(I_0)$.

For the following results let $X = \{\phi : A\phi \equiv \phi - \triangle\phi \in L^p, \partial_\nu \phi = 0\}$ and let X^α be the space $D(A^\alpha)$ endowed with the graph norms $||.||_\alpha$ of A^α for $\frac{n}{2p} < \alpha < 1$.

Theorem 3.4. *Suppose that conditions (P_1) and (P_2) are satisfied and $(u_0, \theta_0) \in L^\infty \times X^\alpha$. Then the solution $(u, \theta) \in L^\infty \times X^\alpha$ satisfies*

$$\sup_{0 \leq t < \infty} ||\theta(t)||_{X^\alpha} \leq C_1(||\theta_0||_{X^\alpha}, ||u_0||_\infty)$$

$$\sup_{0 \leq t < \infty} ||u(t)||_\infty \leq C_2(||\theta_0||_{X^\alpha}, ||u_0||_\infty)$$

$$\lim_{t \to \infty} ||\theta - \bar{\theta}||_{W^{1,q}} = \lim_{t \to \infty} ||u_t||_2 = 0,$$

for some $q > n$.

If $u_0(x) \in W^{1,\sigma}(\Omega)$, $\theta_0(x) \in W^{2,\sigma}(\Omega)$ for $\sigma > n$, and $f' + \int_\Omega J(x-y)dy \geq C > 0$, then there exists a subsequence t_k such that

$$(u(t_k), \theta(t_k)) \to (u^*, c) \text{ in } C^\gamma(\Omega),$$

where (u^*, c) is a steady state solution.

4. Pulses for the Nonlocal Wave Equation

The results here are with Chunlei Zhang and are given in full in [20]. We consider the nonlocal wave equation for $u(x,t)$:

$$(4.1) \quad u_{tt} - \frac{1}{\varepsilon^2}(j_\varepsilon * u - u) + f(u) = 0, \quad \text{for} \quad t > 0 \quad \text{and} \quad x \in \mathbb{R},$$

where ε is a positive parameter and the kernel j_ε of the convolution is defined by

$$j_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon}),$$

where $j(\cdot)$ is an even function with unit integral. We assume that f is a C^2 function, satisfying $f(0) = 0$ and $f(\zeta_0) > 0$, where

$$\zeta_0 = \inf\{\zeta > 0 : F(\zeta) = 0\} \text{ and } F(\zeta) = \int_0^\zeta f(s)ds.$$

Typical examples include the quadratic function $f(u) = u(u-a)$ or the cubic $f(u) = u(u-b)(u+c)$ with $a, b,\ c > 0$.

We also consider a lattice version

$$\ddot{u}_n - \frac{1}{\varepsilon^2}\sum_{k=-\infty}^{\infty} \alpha_k u_{n-k} + f(u_n) = 0, \qquad n \in \mathbb{Z}. \tag{4.2}$$

Note that, as $\varepsilon \to 0$, $\frac{1}{\varepsilon^2}(j_\varepsilon * u - u) \to du_{xx}$, formally and in some weak sense described in [10], where d is a constant determined by j. So we can also regard (4.1) as a nonlocal version of the standard nonlinear wave equation

$$u_{tt} - du_{xx} + f(u) = 0. \tag{4.3}$$

In this paper we will study homoclinic traveling wave solutions of (4.1), i.e., solutions of the form $u(x,t) = u(x-ct)$ which decay at infinity.

It is worth mentioning here that the parabolic versions

$$u_t - \frac{1}{\varepsilon^2}(j_\varepsilon * u - u) + f(u) = 0, \quad \text{for} \quad t > 0 \quad \text{and} \quad x \in \mathbb{R}, \tag{4.4}$$

and

$$\dot{u}_n - \frac{1}{\varepsilon^2}\sum_{k=-\infty}^{\infty} \alpha_k u_{n-k} + f(u_n) = 0, \qquad n \in \mathbb{Z}, \tag{4.5}$$

where f is bistable (e.g., the cubic above) were treated in [10] and [9], respectively, where traveling or stationary waves were shown to exist, connecting the stable zeros of f. Certain assumptions are needed upon j and the α_k's but we note that they are not required to be non-negative, i.e., they may change sign. When the wave has nonzero velocity, then the results in those papers are perturbative and rely upon spectral theory that we develop for the linearized operators for $\varepsilon > 0$ sufficiently small. When the wave is stationary (the potential has wells of equal depth), then under the conditions imposed on the coefficients, solutions exist for all $\varepsilon > 0$.

In this paper, we make slightly different assumptions on j and the α_k's than in [10] and [9], and the proofs are very different, but in some sense spectral analysis is still involved. To be more precise, we assume

(W_1) $f \in C^2(\mathbb{R}), f(0) = 0, f'(0) = -a < 0;$ $f(\zeta_0) > 0$, where

$$\zeta_0 \equiv \inf\{\zeta > 0 : F(\zeta) = 0\} \text{ and } F(\zeta) = \int_0^\zeta f(s)ds.$$

(W_2) $j(x) \in L^1(R)$ is even, has unit integral,

$$\lim_{z\to 0} \frac{\widehat{j}(z) - 1}{z^2} = -d \qquad \text{and} \qquad \widehat{j}(z) \geq 1 - d_1 z^2,$$

where $0 < d \leq d_1$ are constants and the Fourier transform is given by $\widehat{j}(z) \equiv \int_{-\infty}^{\infty} e^{-izx} j(x)dx$.

Remark 4.1. If $\widehat{j} \in C^2$ then $d = \frac{1}{2}\int_{-\infty}^{\infty} j(x)x^2dx$.

In (4.1), let $u(x,t) = u(x-ct) = u(\eta)$, so that $u(\eta)$ satisfies the equation

$$c^2u'' - \frac{1}{\varepsilon^2}(j_\varepsilon * u - u) = -g(u) + au, \tag{4.6}$$

where $g(u) = f(u) + au$. Applying the Fourier transform, equation (4.6) becomes

$$-c^2\xi^2\hat{u} - \frac{1}{\varepsilon^2}(\widehat{j_\varepsilon} \cdot \hat{u} - \hat{u}) = -\widehat{g(u)} + a\hat{u}$$

or

$$(c^2\xi^2 + l_\varepsilon(\xi) + a)\hat{u} = \widehat{g(u)},$$

where $l_\varepsilon = \frac{1}{\varepsilon^2}(\widehat{j_\varepsilon} - 1)$. Thus, an equivalent formulation is

$$\hat{u} = p_\varepsilon(\xi)\widehat{g(u)},$$

where

$$p_\varepsilon(\xi) = \frac{1}{c^2\xi^2 + l_\varepsilon(\xi) + a}. \tag{4.7}$$

The inverse Fourier transform gives

$$u = \check{p}_\varepsilon * g(u), \tag{4.8}$$

where $\check{p}_\varepsilon$ is the inverse transform of p_ε.

Define the operator

$$P_\varepsilon(u) \equiv \check{p}_\varepsilon * g(u),$$

and write (4.8) as

$$u = P_\varepsilon(u). \tag{4.9}$$

Note that, due to (W_2),

$$l_\varepsilon(\xi) = \frac{1}{\varepsilon^2}(\widehat{j_\varepsilon}(\xi) - 1) = \frac{\widehat{j}(\varepsilon\xi) - 1}{(\varepsilon\xi)^2} \cdot \xi^2 \to -d\xi^2$$

as $\varepsilon \to 0$, so $p_\varepsilon \approx \dfrac{1}{(c^2-d)\xi^2 + a}$ for ε small. Thus, when $\varepsilon \to 0$, (4.9) formally becomes

$$u = P_0(u), \tag{4.10}$$

where $P_0(u) \equiv \check{p}_0 * g(u)$ and

$$p_0(\xi) = \frac{1}{(c^2-d)\xi^2 + a}.$$

Clearly, (4.10) is equivalent to

$$u = ((d - c^2)\partial^2 + a)^{-1}(g(u)),$$

that is,

$$(c^2 - d)u'' = au - g(u),$$

or

$$(c^2 - d)u'' + f(u) = 0. \tag{4.11}$$

By the results in [21], under the assumption (W_1), (4.11) has a unique even, positive homoclinic solution for each $c^2 > d$, which we denote by u_0. Thus, u_0 is a fixed point of operator P_0. We can write equation (4.6) in the form

$$u = P_0(u) + (P_\varepsilon - P_0)(u),$$

and look for a fixed point near u_0.

In the case of the lattice Klein-Gordon equation (4.2), we assume

(W_3)

$$\sum_{k=-\infty}^{\infty} \alpha_k = 0, \quad \alpha_0 < 0, \quad \alpha_k = \alpha_{-k}, \quad \sum_{k\geq 1} \alpha_k k^2 = d > 0$$

$$\text{and} \quad \sum_{k\geq 1} |\alpha_k| k^2 = \bar{d} < \infty.$$

With the ansatz $u_n(t) = u(\varepsilon n - ct) = u(\eta)$, we get the following differential equation with infinitely many advanced and delayed terms:

$$c^2 \frac{d^2 u}{d\eta^2} - \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k u(\eta - k\varepsilon) + f(u) = 0. \tag{4.12}$$

Applying the Fourier transform, equation (4.12) becomes

$$-c^2 \xi^2 \hat{u} - \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k e^{i\varepsilon k\xi} \hat{u} + \widehat{f(u)} = 0.$$

Using (W_3) we may write

$$\frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k e^{i\varepsilon k\xi} \hat{u} = \frac{1}{\varepsilon^2} \sum_{k\geq 1} \alpha_k (e^{i\varepsilon k\xi} - 2 + e^{-i\varepsilon k\xi})\hat{u}$$

$$= \frac{4}{\varepsilon^2} \sum_{k\geq 1} \alpha_k \sin^2(\frac{\varepsilon k\xi}{2})\hat{u}.$$

Therefore, we may write our equation as

$$[(c^2 - \sum_{k\geq 1} \alpha_k k^2 \mathrm{sinc}^2(\frac{\varepsilon k\xi}{2}))\xi^2 + a]\hat{u} = \widehat{g(u)},$$

where $\mathrm{sinc}(z) = \dfrac{\sin z}{z}$.

We define

$$D_1 =: \sup_z \sum_{k\geq 1} \alpha_k k^2 \mathrm{sinc}^2(kz)$$

and note that $D_1 \geq d$.

Let

$$q_\varepsilon(\xi) = \frac{1}{(c^2 - \sum_{k\geq 1} \alpha_k k^2 \mathrm{sinc}^2(\frac{\varepsilon k \xi}{2}))\xi^2 + a},$$

then we can write the equation in the form

$$u = Q_\varepsilon(u),$$

where Q_ε is the operator defined by $Q_\varepsilon(u) =: \check{q}_\varepsilon * g(u)$. Since

$$\mathrm{sinc}^2(\frac{\varepsilon k \xi}{2}) = 1 - \frac{1}{8}\varepsilon^2\xi^2 + o(\varepsilon^4\xi^4),$$

$q_\varepsilon(\xi)$ has the limit

$$p_0(\xi) = \frac{1}{(c^2 - d)\xi^2 + a}$$

as $\varepsilon \to 0$. Therefore, formally when $\varepsilon \to 0$, we have the limit equation

$$u = P_0(u),$$

where, as before, $P_0(u) =: \check{p}_0 * g(u)$; and the integral equation is equivalent to (4.11), which has the nondegenerate homoclinic solution, u_0, discussed previously.

This time we have the fixed point problem

$$u = P_0(u) + (Q_\varepsilon - P_0)(u),$$

and the idea is to show that $Q_\varepsilon - P_0$ (or $P_\varepsilon - P_0$) in the previous case) is sufficiently small in some neighborhood of u_0 that a fixed point exists. An abstract lemma in [51], from which we borrowed this plan of attack, gives conditions under which this is the case. Basically, the lemma is a variant of the Implicit Function Theorem. The conditions are essentially that P_0 and P_ε (or Q_ε) are close as C^1 mappings on a ball about u_0 in a Banach space, with $I - DP(u_0)$ invertible with small norm.

Armed with this lemma we are able to prove the following theorem by showing that the various hypotheses hold.

Theorem 4.2. *Under the assumptions* (W_1) *and* (W_2) *(or* (W_3)*), there exits an* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and speed* c *satisfying* $c^2 > d_1$*, (or* $c^2 > D_1$*) equation* (4.6) *(or* (4.12)*) has a unique nonzero solution* u_ε *in the set*

$$\{u \in H^1(\mathbb{R}) :\ u \text{ is even},\ \|u - u_0\|_{H^1} < \delta\},$$

where u_0 *is the even, positive homoclinic solution of* (4.11) *and* $\delta > 0$ *depends on* j *and* f *and satisfies* $\delta < \|u_0\|_{H^1}$.

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